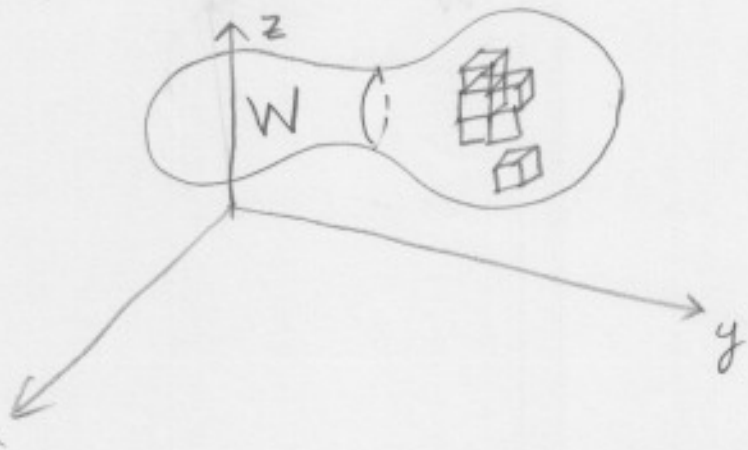
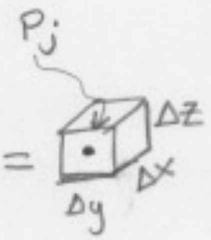
Section 5.5
Triple integral

Consider an open set W in $\mathbb{R}^3$. Suppose that $f:\mathbb{R}^3 \to \mathbb{R}$ is a function that gives the density (say in gr/cm^3) of the body W at the point $(x,y,z) \in W$. Assume also that $f(x,y,z)$ is a continuous function.

<u>Question</u>: Compute the mass of W.

As usual we compute the Riemann sums S_n by splitting our domain W in to small rectangles $Q_1, Q_2, \dots, Q_n$.



$Q_j =$ 

The volume of Q_j is $\Delta x \Delta y \Delta z$ and we choose a point $P_j \in Q_j$.

The mass of W is approximated by:

$$M \cong S_n = \sum_{j=1}^{n} f(P_j)\Delta x \, \Delta y \, \Delta z$$

$$gr/\cancel{cm^3} \cdot \cancel{cm^3} = gr.$$

We now let $n \to \infty$ (i.e., $\Delta x, \Delta y, \Delta z \to 0$) to get the exact mass of W:

$$M = \lim_{\substack{n \to \infty \\ (\Delta x, \Delta y, \Delta z \to 0)}} S_n$$

$$= \lim_{n \to \infty} \sum_{j=1}^{n} f(P_j)\Delta x \Delta y \Delta z$$

Theorem: Since f is continuous the above limit exits, say S. Moreover, the limit is independent of the choice of points P_j. Hence, the mass of W is S and we denote S as $\iiint_W f\,dV$.

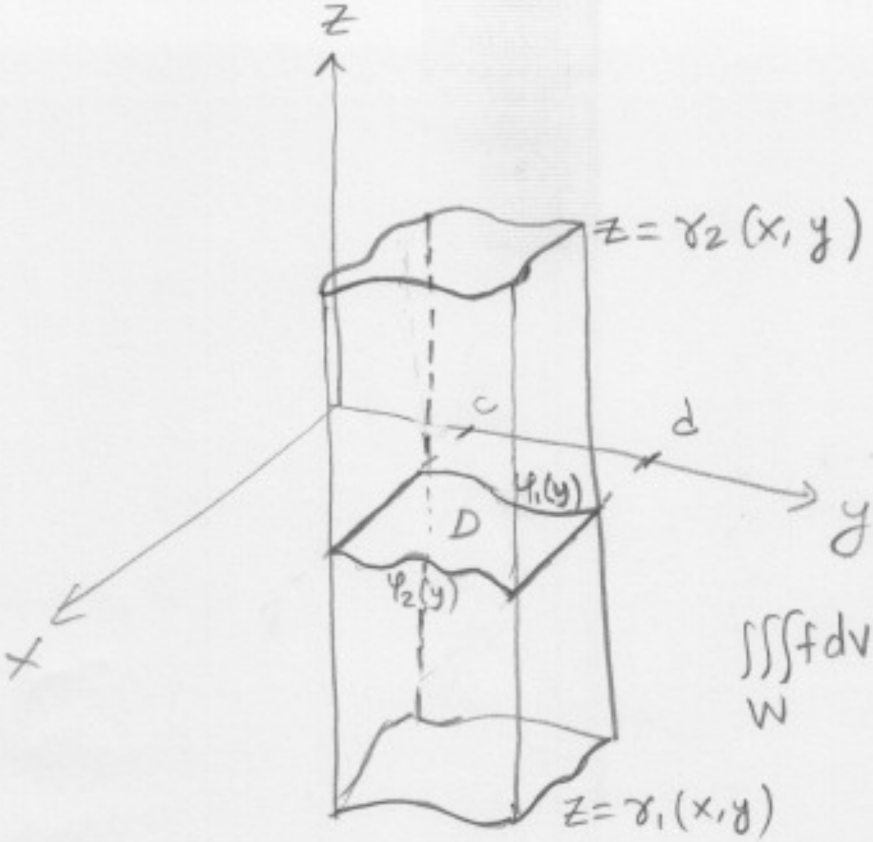
How do we compute S in practice?
If W is an elementary region (defined below) we can use Fubini's theorem to compute it. If W is not an elementary region, we could brake it in to parts, where each part is an elementary region.

Definition: W is an elementary region if it is of the form;

$$W = \{(x,y,z) \in \mathbb{R}^3 : a \le x \le b, \varphi_1(x) \le y \le \varphi_2(x), \gamma_1(x,y) \le z \le \gamma_2(x,y)\}$$

or

$$W = \{(x,y,z) \in \mathbb{R}^3 : c \le y \le d, \varphi_1(y) \le x \le \varphi_2(y), \gamma_1(x,y) \le z \le \gamma_2(x,y)\}$$

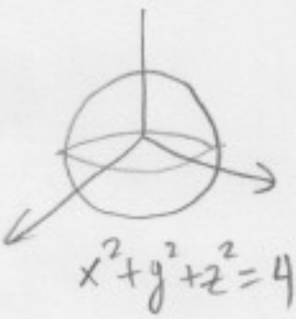


$$\iiint_W f\, dV = \int_c^d \int_{\varphi_1(y)}^{\varphi_2(y)} \int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x,y,z)\, dz\, dx\, dy$$

<u>Note</u>: If $f(x,y,z) \equiv 1$ then instead of mass, we are computing the volume of W.

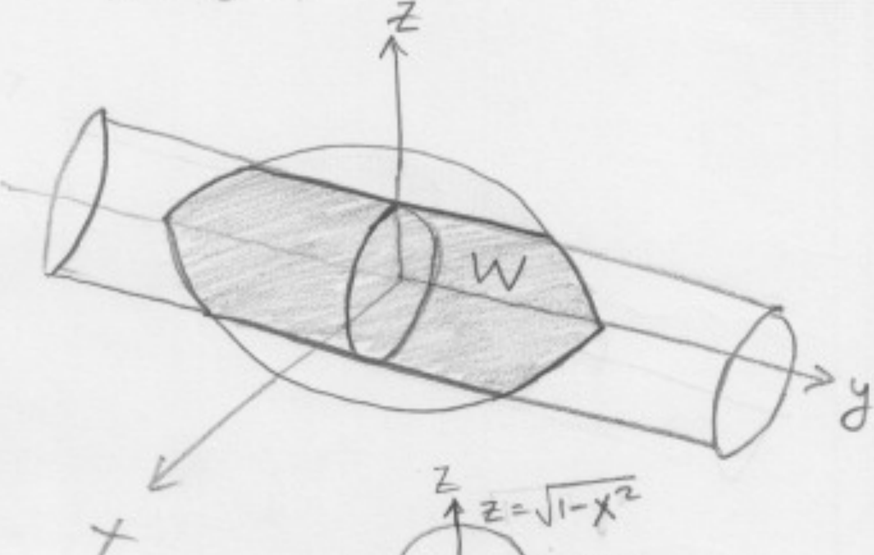
$$V(W) = \iiint_W 1 \cdot dx\, dy\, dz$$

<u>Ex</u>: Let B be the ball centered at the origin, with radius 2. Write an iterated integral for $\iiint_B f(x,y,z)\, dV$.



$$\iiint_B f(x,y,z)\, dV = \int_{\underset{a}{-2}}^{\overset{b}{2}} \int_{\underset{\varphi_1(x)}{-\sqrt{4-x^2}}}^{\overset{\varphi_2(x)}{\sqrt{4-x^2}}} \int_{\underset{\gamma_1(x,y)}{-\sqrt{4-x^2-y^2}}}^{\overset{\gamma_2(x,y)}{\sqrt{4-x^2-y^2}}} f(x,y,z)\, dz\, dy\, dx$$

Ex: W is the region cut out of ellipsoid $x^2+y^2+2z^2=4$ by the cylinder $x^2+z^2=1$

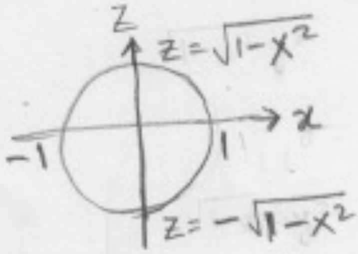


$x=0 \Rightarrow y^2+2z^2=4$

$y^2+\frac{z^2}{2}=1$

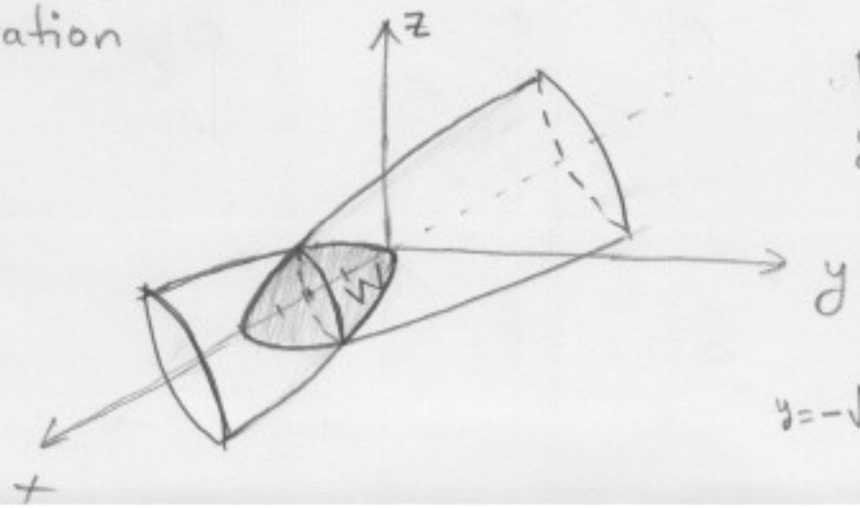
$y=0 \Rightarrow x^2+2z^2=4$

$\frac{x^2}{4}+\frac{z^2}{2}=1$



$$\iiint_W f(x,y,z)\,dV = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{4-x^2-2z^2}}^{\sqrt{4-x^2-2z^2}} f(x,y,z)\,dy\,dz\,dx$$

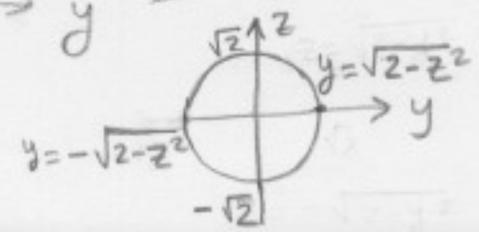
Ex: Find the volume of the region W bounded by $x=y^2+z^2$ and $x=4-y^2-z^2$ by triple integration



Find intersection:

$y^2+z^2=4-y^2-z^2$

$\boxed{y^2+z^2=2,\ x=2}$



$$\iiint_W 1 \, dV = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-z^2}}^{\sqrt{2-z^2}} \int_{y^2+z^2}^{4-y^2-z^2} 1 \cdot dx \, dy \, dz$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-z^2}}^{\sqrt{2-z^2}} \left(4 - 2y^2 - 2z^2\right) dy \, dz$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \left[4y - \frac{2}{3}y^3 - 2z^2 y\right]_{-\sqrt{2-z^2}}^{\sqrt{2-z^2}} dz$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \Big(4\sqrt{2-z^2} - \frac{2}{3}(2-z^2)^{3/2} - 2z^2\sqrt{2-z^2} + 4\sqrt{2-z^2}$$
$$- \frac{2}{3}(2-z^2)^{3/2} - 2z^2\sqrt{2-z^2}\Big) dz$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \left(8\sqrt{2-z^2} - \frac{4}{3}(2-z^2)^{3/2} - 4z^2\sqrt{2-z^2}\right) dz$$

Change of variables: $z = \sqrt{2}\sin\theta$, $dz = \sqrt{2}\cos\theta \, d\theta$

$\sin\theta = -1$ $\qquad$ $\sin\theta = 1$

$\theta = -\frac{\pi}{2}$ $\qquad$ $\theta = \frac{\pi}{2}$

(or $\frac{3}{2}\pi$) $\qquad$ (or $\frac{5}{2}\pi$)

$$= 8\int_{-\pi/2}^{\pi/2} \sqrt{2 - 2\sin^2\theta}\,\sqrt{2}\cos\theta \, d\theta - \frac{4}{3}\int_{-\pi/2}^{\pi/2} (2 - 2\sin^2\theta)^{3/2}\sqrt{2}\cos\theta \, d\theta$$

$$- 4\int_{-\pi/2}^{\pi/2} 2\sin^2\theta \sqrt{2 - 2\sin^2\theta}\,\sqrt{2}\cos\theta \, d\theta$$

$$= 16\int_{-\pi/2}^{\pi/2} \cos^2\theta \, d\theta - \frac{16}{3}\int_{-\pi/2}^{\pi/2} \cos^4\theta \, d\theta - 16\int_{-\pi/2}^{\pi/2} \underbrace{\sin^2\theta}_{1-\cos^2\theta} \cos^2\theta \, d\theta$$

$$16 - \frac{16}{3} = \frac{48-16}{3} = \frac{32}{2} = 1$$

$$= \frac{32}{3}\int_{-\pi/2}^{\pi/2} \cos^4\theta \, d\theta$$

$$= \frac{32}{3}\int_{-\pi/2}^{\pi/2} \left(\frac{3}{8} + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta\right) d\theta$$

$$= \frac{32}{3}\left[\frac{3}{8}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{32}{\cancel{3}} \cdot \frac{\cancel{3}}{8}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \boxed{4\pi}$$

Trigonometric identities used above:

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= \cos^2\theta - (1 - \cos^2\theta) = 2\cos^2\theta - 1$$

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$\cos^4\theta = \frac{1}{4}(1 + 2\cos 2\theta + \cos^2 2\theta)$$
$$= \frac{1}{4} + \frac{1}{2}\cos 2\theta + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}\cos 4\theta\right)$$
$$= \frac{1}{4} + \frac{1}{2}\cos 2\theta + \frac{1}{8} + \frac{1}{8}\cos 4\theta$$
$$= \frac{3}{8} + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta$$