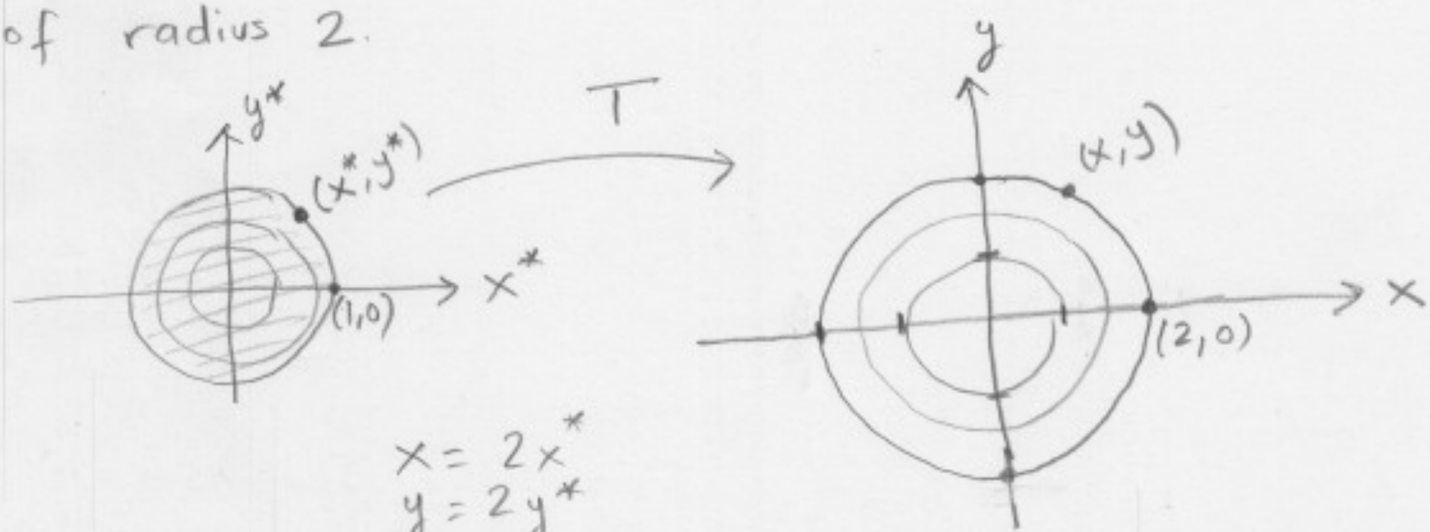
Section 6.1

Geometry of maps $\mathbb{R}^2 \to \mathbb{R}^2$.

Let $D^* \subseteq \mathbb{R}^2$ and $T: D^* \to \mathbb{R}^2$. We consider the image $T(D^*)$ of D^* under T.

<u>Ex</u>: Let D^* be the disk $(x^*)^2+(y^*)^2 \le 1$, and $T(x^*, y^*) = (2x^*, 2y^*)$. Then $T(D^*)$ is the disk of radius 2.



$$x = 2x^*$$
$$y = 2y^*$$

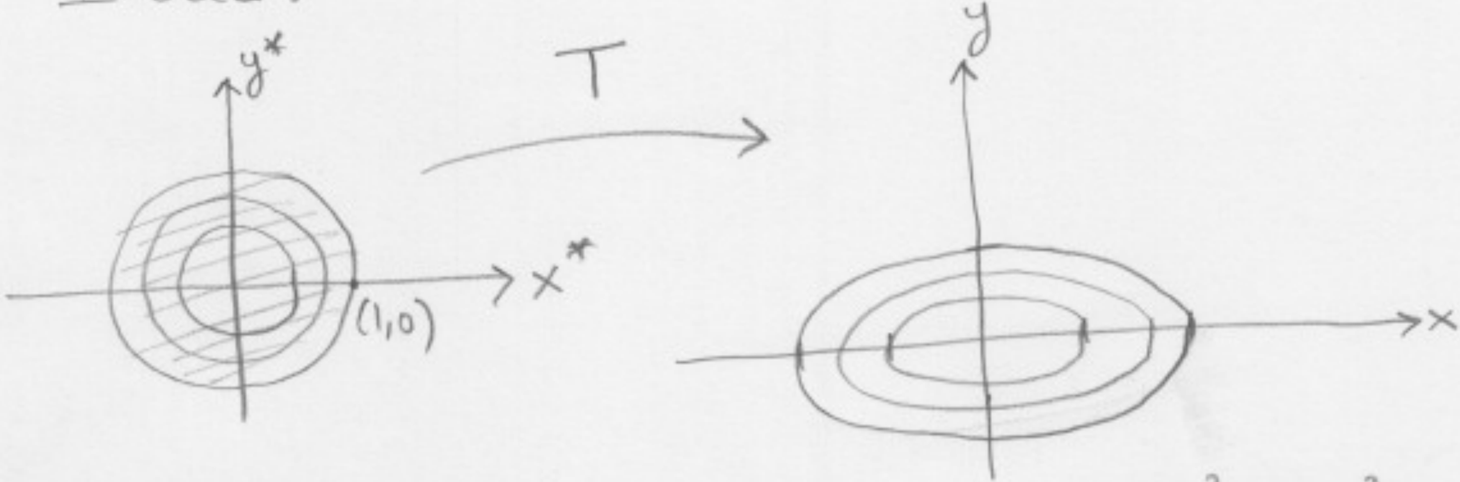
If $(x^*)^2+(y^*)^2=1$ then $x^2+y^2 = 4(x^{*2}+y^{*2}) = 4(1) = 4$, so T maps the circle $(x^*)^2+(y^*)^2=1$ to the circle $x^2+y^2=4$, T maps the circle $(x^*)^2+(y^*)^2=\frac{1}{2}$ to the circle $x^2+y^2=1$, and so on.

<u>Definition</u>: We say that T is a one to one map if for every point (x,y) in $T(D^*)$ there is just one point $(x^*,y^*) \in D^*$ such that

$$T(x^*, y^*) = (x,y)$$

Clearly, the previous map T is one to one.

<u>Ex</u>: Let again $D^* = \{(x^*, y^*) : (x^*)^2 + (y^*)^2 \leq 1\}$, and $T(x^*, y^*) = (2x^*, y^*)$. Then T maps the disk D^* in to the ellipses $\frac{x^2}{4} + y^2 \leq 1$. Indeed:

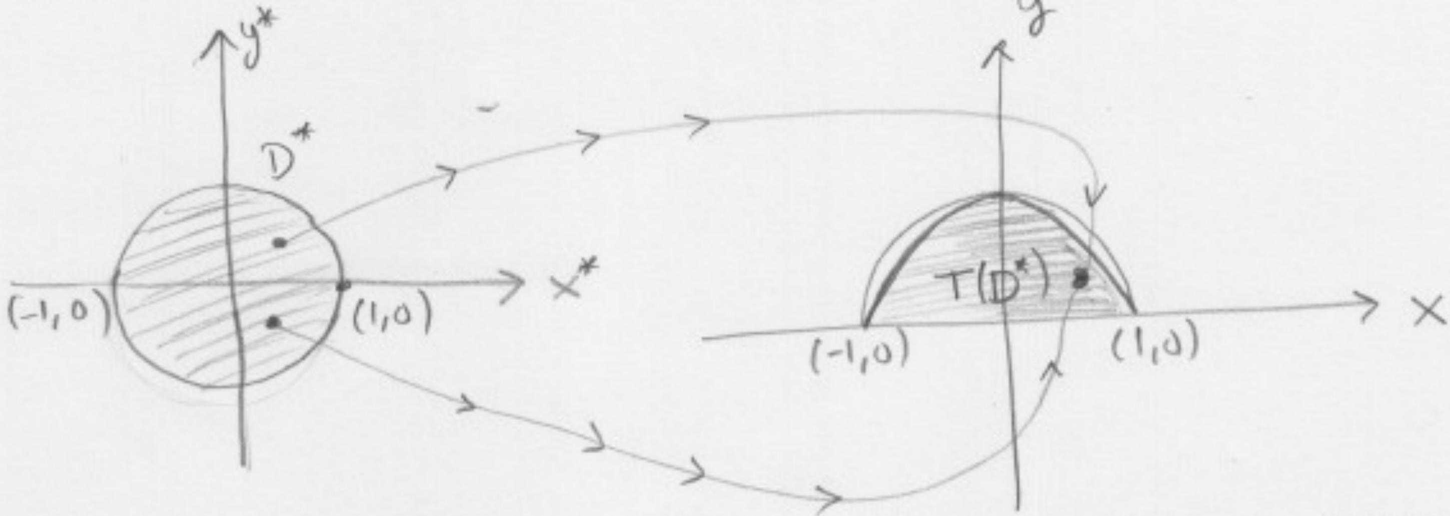


We have $x = 2x^*$, $y = y^*$, so if $(x^*)^2 + (y^*)^2 = 1$ we have $\frac{x^2}{4} + y^2 = (x^*)^2 + (y^*)^2 = 1$. Hence, T maps the circle $(x^*)^2 + (y^*)^2 = 1$ in to the ellipse $\frac{x^2}{4} + y^2 = 1$. Also, T maps the circle $(x^*)^2 + (y^*)^2 = \frac{1}{2}$ in to the inner ellipse $x^2 + \frac{y^2}{1/4} = 1$, and so on.

We think of T as deforming the disk $(x^*)^2 + (y^*)^2 \leq 1$ in to the ellipses $\frac{x^2}{4} + y^2 \leq 1$.

Notice that T is also one to one.

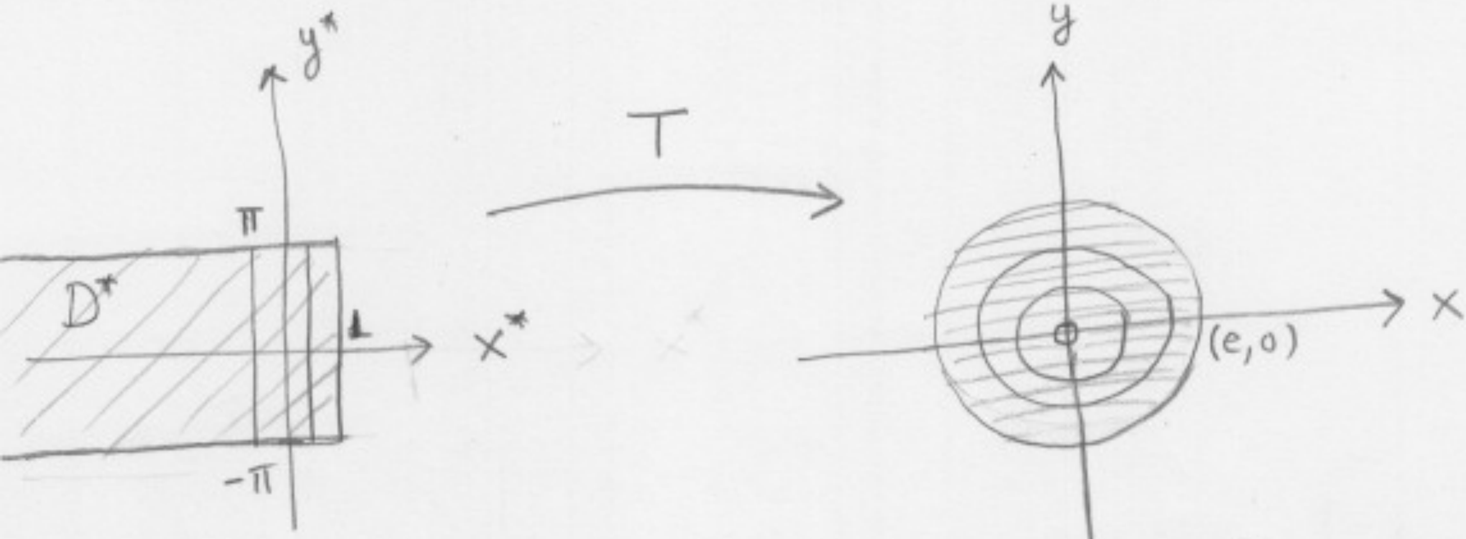
Ex: $T(x^*, y^*) = (x^*, y^{*2})$ with D^* as before. In this case $T(D^*)$ is contained inside the upper half disk. Also T is not 1-1 (one to one). It is in fact two to one.



We have $x = x^*$ and $y = (y^*)^2$, hence $y^* = \pm\sqrt{y}$ and $(x^*)^2 + (y^*)^2 = \underline{1}$ implies $x^2 + y = 1$, or $y = 1 - x^2$. The image of D^* is the inside of the parabola $y = 1 - x^2$, in the upper half disk.

Notice that this map T is <u>NOT</u> one-to one, since two points in D^* are mapped to the same point in $T(D^*)$.

Ex: Let $T(x^*, y^*) = (e^{x^*}\cos y^*, e^{x^*}\sin y^*)$ where D^* is as follows:



$$D^* = \{(x^*, y^*) : -\infty < x^* \le 1, -\pi \le y^* \le \pi\}$$

We have $x = e^{x^*}\cos y^*$, $y = e^{x^*}\sin y^*$, so $x^2+y^2 = (e^{x^*})^2 = e^{2(x^*)^2}$. Hence, if $x^* = 1$ we obtain $x^* + y^* = e^2$, the circle of radius e. Every vertical line in D^* is mapped to a circle in $T(D^*)$. Note that the origin is never attained, since x, y can never be both zero.

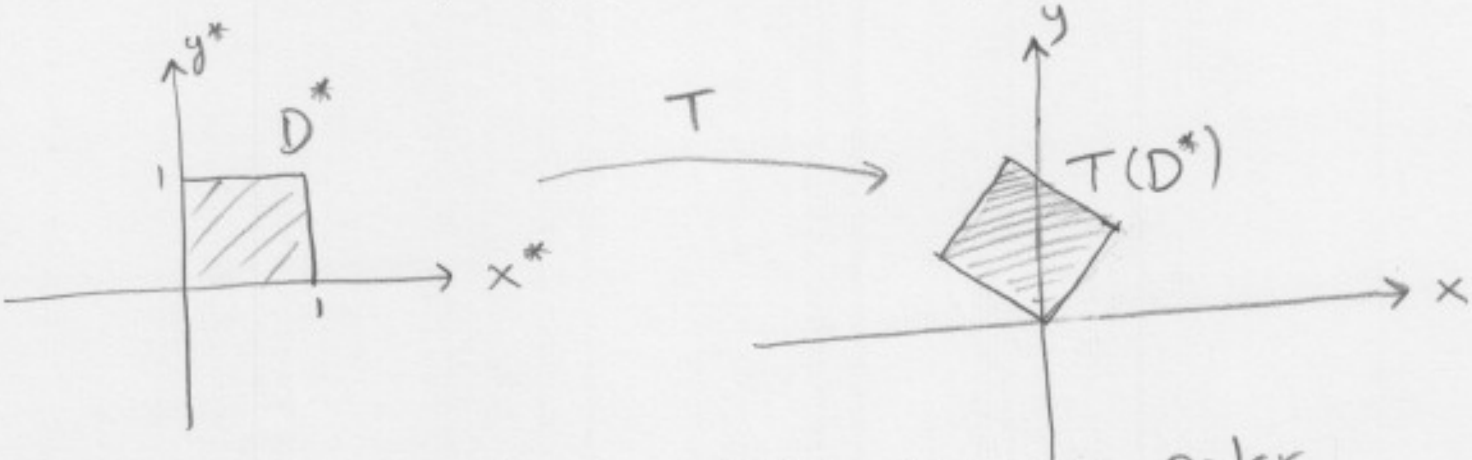
T is not one to one, since for example $T(x^*, \pi) = T(x^*, -\pi)$. T becomes one to one if we take:

$$D^* = \{(x^*, y^*) : -\infty < x^* \le 1, -\pi < y^* \le \pi\}$$

Ex: Let $D^* = [0,1] \times [0,1]$

$$T(x^*, y^*) = \left(\frac{x^* - y^*}{\sqrt{2}}, \frac{x^* + y^*}{\sqrt{2}} \right)$$

Then T rotates the square:



In order to see this we use polar coordinates. Let:

$$x^* = r\cos\theta \qquad y^* = r\sin\theta$$

$$\Rightarrow T(x^*, y^*) = T(r\cos\theta, r\sin\theta)$$

$$= \left(\frac{r\cos\theta}{\sqrt{2}} - \frac{r\sin\theta}{\sqrt{2}}, \frac{r\cos\theta}{\sqrt{2}} + \frac{r\sin\theta}{\sqrt{2}} \right)$$

$$= \left(r\cos\theta\cos\tfrac{\pi}{4} - r\sin\theta\sin\tfrac{\pi}{4}, r\cos\theta\sin\tfrac{\pi}{4} + r\sin\theta\cos\tfrac{\pi}{4} \right)$$

$$= \left(r\cos\left(\theta + \tfrac{\pi}{4}\right), r\sin\left(\theta + \tfrac{\pi}{4}\right) \right)$$

which is the original point rotated $\frac{\pi}{4}$ counterclockwise.

Ex: Let $T(u,v) = (-u^2+4u, v)$ $D^* = [0,1]\times[0,1]$

Is T one to one?

Suppose $T(u_1,v_1) = T(u_2,v_2)$. We need to show that $u_1 = u_2$ and $v_1 = v_2$.

$$\Rightarrow (-u_1^2+4u_1, v_1) = (-u_2^2+4u_2, v_2)$$

$$\Rightarrow \boxed{v_1 = v_2}$$

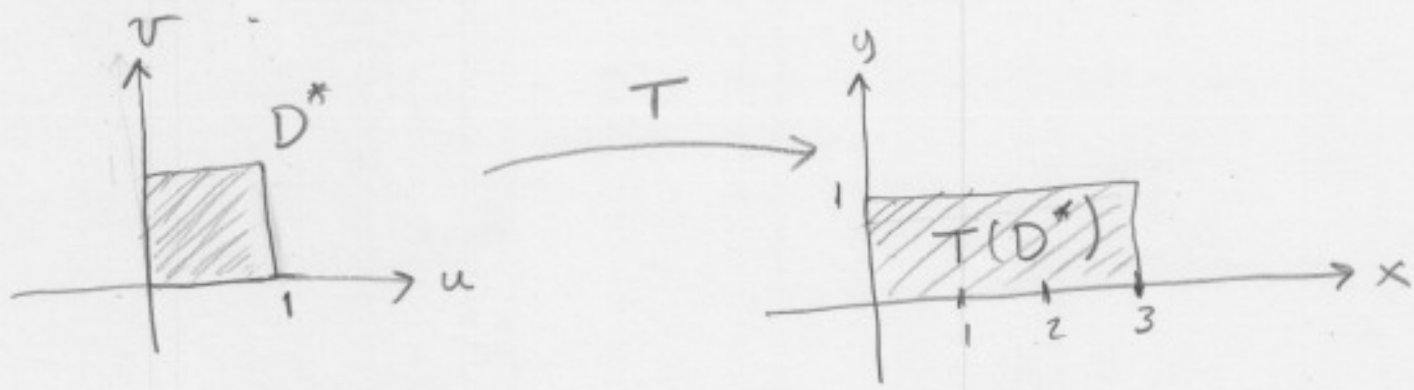
$$\Rightarrow -u_1^2+4u_1 = -u_2^2+4u_2 \longrightarrow (*)$$

Let $f(u) = -u^2+4u$

$f'(u) = -2u+4 > 0$ if $0 \le u \le 1$.

Since f is increasing strictly on $[0,1]$, then $f(u_1) \neq f(u_2)$ if $u_1 \neq u_2$. Therefore, from $(*)$ we conclude that $\boxed{u_1 = u_2}$.

We conclude that T is one to one



$T(0,0) = (0,0)$ $T(0,1) = (0,1)$

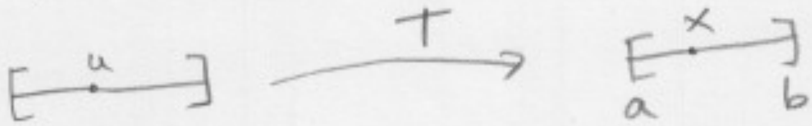
$T(1,0) = (3,0)$ $T(1,1) = (3,1)$

To get the image $T(D)$ we need to look only at the image of the boundary

The reason why we want to "deform" domains in to help with integration. If we have $T(D^*) = D$ and we want to compute $\iint_D f(x,y)\, dA$, and the domain D is more complex, we may want to integrate instead on D^*. We are familiar with this procedure in 1-variable:

Change of variables formula in 1-d.

In elementary calculus we change variables (make substitutions) as follows:



$D^* = [c,d]$ $\qquad T(D^*) = [a,b]$

$$T(u) = x(u)$$

$$x(c) = a, \quad x(d) = b, \qquad dx = \frac{dx}{du} \cdot du$$

Then:

$$\int_a^b f(x)\, dx = \int_c^d f(x(u)) \frac{dx}{du} \cdot du.$$

In higher dimensions we use the Jacobian determinant as we will see next.