RRT 2: Sketch of proof:

Assume first $\mu(X) < \infty$.

Let $E \in \mathcal{M} \Rightarrow \mu(E) < \infty \Rightarrow \chi_E \in L^p(X)$

Define:

$$\nu(E) = F(\chi_E), \quad E \in \mathcal{M}.$$

<u>ν is a signed measure</u>:

Suppose $\{E_k\}$ is a sequence of disjoint measurable sets. Let $F \in (L^p(X))^*$. Let:

$$E := \bigcup_{k=1}^{\infty} E_k$$

$$\left|\nu(E) - \sum_{k=1}^{N} \nu(E_k)\right| = \left|F(\chi_E) - F\left(\sum_{k=1}^{N} \chi_{E_k}\right)\right|$$

$$= \left|F\left(\chi_E - \sum_{k=1}^{N} \chi_{E_k}\right)\right|$$

$$= \left|F\left(\sum_{k=N+1}^{\infty} \chi_{E_k}\right)\right|$$

$$\leq \|F\| \left\|\sum_{k=N+1}^{\infty} \chi_{E_k}\right\|_p$$

$$= \|F\| \left(\int_X \left|\sum_{k=N+1}^{\infty} \chi_{E_k}\right|^p\right)^{1/p}$$

$$= \|F\| \left(\int_X \chi_{\bigcup_{N+1}^{\infty} E_k} \, d\mu\right)^{1/p}$$

$$= \|F\| \left(\int_{\bigcup_{K=N+1}^{\infty} E_K} d\mu \right)^{1/p}$$

$$= \|F\| \left(\mu \left(\bigcup_{K=N+1}^{\infty} E_K \right) \right)^{1/p}$$

and

$$\mu \left(\bigcup_{K=N+1}^{\infty} E_K \right) = \sum_{K=N+1}^{\infty} \mu(E_K) \to 0 \text{ as } N \to \infty$$

because

$$\mu(E) = \sum_{K=1}^{\infty} \mu(E_K) < \infty$$

$\therefore$ We have shown, $\forall \varepsilon > 0 \ \exists N$ s.t

$$\left| \nu(E) - \sum_{K=1}^{N} \nu(E_K) \right| < \varepsilon \qquad \forall K \geq N.$$

and that implies:

$$\boxed{\nu(E) = \sum_{K=1}^{\infty} \nu(E_K)}$$

$\underline{\nu \ll \mu}$ (ν is absolutely continuous with respe to to μ):

This follows from:

$$|\nu(E)| = |F(\chi_E)| \leq \|F\| \, \|\chi_E\|_p = \|F\| \mu(E)^{1/p}$$

i.e. $\quad |\nu(E)| \leq \|F\| \mu(E)^{1/p}$

Then

$\mu(E) = 0 \Rightarrow \nu(E) = 0.$

By Radon-Nikodym Theorem:

$\exists\, g \in L^1(X)$ such that.

$\nu(E) = F(\chi_E) = \int_E g \, d\mu = \int_X \chi_E g \, d\mu.$

$\therefore$

$$F(\chi_E) = \int_X \chi_E g \, d\mu, \quad \forall \text{ simple function } \chi_E$$

$\Rightarrow$

(A) $F(f) = \int_X f g \, d\mu, \quad \forall$ simple function $\sum_{i=1}^{N} \chi_{E_i}$

$\underline{1}$: $1 \le p < \infty$, $\mu(X) < \infty$

We need to show $g \in L^{p'}(X)$ (See Theorem 183.1). Also:

$\forall f \in L^p(X)\ \exists \{f_k\}$ simple functions s.t:

$$\|f - f_k\|_p \to 0$$

$$\left| F(f) - \int_X fg \, d\mu \right|$$

$$\leq |F(f-f_k)| + \left| F(f_k) - \int_X fg \, d\mu \right|$$

$$= |F(f-f_k)| + \left| \int_X f_k g \, d\mu - \int_X fg \, d\mu \right|, \text{ by (A)}$$

$$= |F(f-f_k)| + \left| \int_X (f_k - f) g \, d\mu \right|$$

$$\leq \|F\| \underbrace{\|f-f_k\|_p}_{<\varepsilon} + \underbrace{\|f_k - f\|_p}_{<\varepsilon} \|g\|_{p'}$$

$$\leq \varepsilon \left(\|F\| + \|g\|_{p'} \right).$$

ε arbitrary gives:

$$F(f) = \int_X fg \, d\mu, \quad \forall f \in L^p(X).$$

Finally, Theorem 165.1 $\Rightarrow$

$$\|g\|_{p'} = \|F\|.$$

<u>Step 2</u>: Remove assumption $\mu(X) < \infty$.
(see Thm 183.1).

Thm 165.1 $\Rightarrow$ g unique.

Def: $(X, \mathcal{M}, \mu)$ measure space.

The signed measure ν
be absolutely continuous with respect to
μ, writen:

$$\nu << \mu$$

if $\mu(E) = 0 \Rightarrow \nu(E) = 0$

Thm 174.1: If ν is a signed measure
on $\mathcal{M}$ then there exists a unique pair
of mutually singular measures ν^+ and
ν^-, at least one of which is finite,
such that:

$$\nu(E) = \nu^+(E) - \nu^-(E)$$

Note: Mutually singular means:
$\exists N$ s.t. $\nu^+(N) = 0 = \nu^-(X \setminus N)$.

Def: $|\nu| := \nu^+ + \nu^-$ is a measure, called
the total variation of ν.

Theorem 308.1

Theorem 179.2 (Radon-Nikodym). 8.73

Let:

$(X, \mathcal{M}, \mu)$ a σ-finite measure space

ν is a σ-finite signed measure on $\mathcal{M}$

$\nu << \mu$.

Then:

$\exists f$ measurable such that either f^+ or f^- is integrable and

$$\nu(E) = \int_E f \, d\mu \quad \forall E \in \mathcal{M}.$$

<u>Proof</u>:

This proof uses RRT1: The Riesz Representation Theorem for Hilbert spaces.

<u>Step 1</u>: Assume $\nu(X) < \infty$, $\mu(X) < \infty$.
ν positive measure.

Define:

$$\boxed{T(f) = \int_X f \, d\nu} \quad \forall f \in L^2(X, \mathcal{M}, \mu+\nu)$$

$$(\mu+\nu)(X) < \infty \Rightarrow L^2(X, \mathcal{M}, \mu+\nu) \subset L^1(X, \mathcal{M}, \mu+\nu)$$

$$\therefore \; T(f) < \infty$$

T is a bounded linear functional because (using Hölder): (8.74)

$$|T(f)| = \left|\int_X f\,d\nu\right|$$

$$\leq \int_X |f|\,d\nu \leq \left(\int_X |f|^2 d\nu\right)^{1/2} (\nu(X))^{1/2}$$

$$\leq \|f\|_{2;\,\nu+\mu}\,(\nu(X))^{1/2}.$$

RRT1 $\Rightarrow \exists\, \varphi \in L^2(\mu+\nu)$ s.t

$$\boxed{T(f) = \int_X f\varphi\, d(\nu+\mu) \quad \forall f \in L^2(\mu+\nu)}$$

<u>Note:</u> $\varphi \geq 0$, $(\mu+\nu)$-a.e (Otherwise $T(\chi_{\{\varphi<0\}}) < 0$,

which is not possible since $T(\chi_{\{\varphi<0\}}) =$

$\int_X \chi_{\{\varphi<0\}}\,d\nu = \int_{\{\varphi<0\}} d\nu = \nu(\{\varphi<0\}) > 0$).

$$\therefore \int_X f\,d\nu = \int_X f\varphi\, d(\nu+\mu) = \int_X f\varphi\,d\nu + \int_X fg\,d\mu$$

$$\therefore \int_X f(1-\varphi)\,d\nu = \int_X f\varphi\,d\mu \quad \forall f \in L^2(\mu+\nu)$$

Let

$f = \chi_E, \quad E = \{\varphi \geq 1\}.$

$$0 \leq \mu(E) = \int_X \chi_E \, d\mu \leq \int_X \chi_E \varphi \, d\mu = \int_X \chi_E (1-\varphi) \, d\nu$$

$$= \int_{\{\varphi \geq 1} (1-\varphi) \, d\nu$$

$$\leq 0$$

$\therefore \quad \mu(E) = 0$

$\therefore \quad \nu(E) = 0 \quad$, since $\nu \ll \mu$.

Let

$g = \varphi \chi_{E^c} \quad , \quad E^c = \{\varphi < 1\}$

$0 \leq \underline{g < 1}, \quad g = \varphi$ a.e with respect to ν and μ

$$(B) \quad \boxed{\int_X f(1-g) \, d\nu = \int_X f g \, d\mu} \quad \forall f \in L^2(\mu+\nu)$$

Replace f by $(1 + g + g^2 + \cdots + g^k) \chi_E$ in (B):

$$\int_X (1+g+\cdots+g^k) \chi_E (1-g) \, d\nu = \int_X (1+\cdots+g^k) \chi_E \, g \, d\mu$$

$\therefore$

$$\int_E \left(1+\cancel{g}+\cdots+g^k-\cancel{g}-g^2-\cdots-g^{k+1}\right) d\nu$$

$$= \int_E g\,(1+g+g^2+\cdots+g^k)\, d\mu$$

$$\therefore \boxed{\int_E (1-g^{k+1})\, d\nu = \int_E g\,(1+g+\cdots+g^k)\, d\mu} \quad (C)$$

$0 \leq g < 1 \implies \sum_{k=1}^{\infty} g^k = h$, h measurable.

From (C) and Monotone Convergence Theorem:

$$\lim_{k\to\infty} \int_E (1-g^{k+1})\, d\nu = \int_E \lim_{k\to\infty} (1-g^{k+1})\, d\nu = \int_E d\nu$$

$$\lim_{k\to\infty} \int_E g\,(1+g+\cdots+g^k)\, d\mu = \int_E \lim_{k\to\infty} g\,(1+g+\cdots+g^k)\, d\mu$$

$$= \int_E \lim_{k\to\infty} (g+g^2+\cdots+g^{k+1}+\cdots)\, d\mu$$

$$= \int_E \sum_{k=1}^{\infty} g^k\, d\mu$$

$$= \int_E h\, d\mu.$$

$$\therefore \boxed{\nu(E) = \int_E h\, d\mu, \quad \forall E \in \mathcal{M}}$$

Step 2: Procced as in Page 180 to consider the case μ and ν σ-finite. Finally, for ν signed, decompose $\nu = \nu^+ - \nu^-$.