- Continuation of the proof of RRT (3).

The last class we defined:

$$\mu(V) := \sup\{L(f) : f \in C_c(\mathbb{R}^n; \mathbb{R}^m), |f| \leq 1, \operatorname{spt}(f) \subset V\},$$

for $V \subset \mathbb{R}^n$ open, and:

$$\mu(A) = \inf\{\mu(V) : A \subset V \text{ open}\},$$

for arbitrary $A \subset \mathbb{R}^n$.

We showed that:

μ is is a <u>Radon outer measure</u>;

that is:

μ is Borel (All Borel sets are μ-measurable),

μ is regular ($\forall A \subset \mathbb{R}^n$, $\exists G$ Borel, $A \subset G$ and $\mu(A) = \mu(G)$.

$\mu(K) < \infty$, $\forall K \subset \mathbb{R}^n$, K compact.

The final goal is to show the representation:

$$L(f) = \int_{\mathbb{R}^n} f \cdot \sigma \, d\mu, \quad \forall f \in C_c(\mathbb{R}^n; \mathbb{R}^m),$$

and some $\sigma : \mathbb{R}^n \to \mathbb{R}^m$, μ-measurable and $|\sigma(x)| = 1$ for μ-a.e. x.

Step 4:

We now define an auxiliary functional:

$$\lambda(f) := \sup\{|L(g)| : g \in C_c(\mathbb{R}^n; \mathbb{R}^m),\ |g| \le f\},$$

for all $f \in C_c^+(\mathbb{R}^n) := \{f \in C_c(\mathbb{R}^n) \mid f \geq 0\}$.

Note:

$f_1, f_2 \in C_c^+(\mathbb{R}^n)$, $\underline{f_1 \le f_2} \Rightarrow \underline{\lambda(f_1) \le \lambda(f_2)}$.

$\underline{\lambda(cf) = c\lambda(f)}$, $c \geq 0$, $f \in C_c^+(\mathbb{R}^n)$

Step 5:

Claim: $\lambda(f_1 + f_2) = \lambda(f_1) + \lambda(f_2)$, for all $f_1, f_2 \in C_c^+(\mathbb{R}^n)$.

Let

$\underline{g_1, g_2 \in C_c(\mathbb{R}^n; \mathbb{R}^m)}$, $\underline{|g_1| \le f_1,\ |g_2| \le f_2}$

$L(g_1), L(g_2) \geq 0$.

$$\Rightarrow |L(g_1)| + |L(g_2)| = L(g_1 + g_2) = |L(g_1 + g_2)| \le \lambda(f_1 + f_2)$$

$|g_1 + g_2| \le f_1 + f_2$

Taking sup over all such g_1, g_2:

$\underline{\lambda(f_1) + \lambda(f_2) \le \lambda(f_1 + f_2)}$.

Fix $g \in C_c(\mathbb{R}^n, \mathbb{R}^m)$,
$|g| \leq f_1 + f_2$.

Let
$$g_i \equiv \begin{cases} \dfrac{f_i g}{f_1+f_2}, & \text{if } f_1+f_2>0 \\ 0, & \text{if } f_1+f_2=0 \end{cases}$$

$g_1, g_2 \in C_c(\mathbb{R}^n, \mathbb{R}^m)$
$g = g_1 + g_2$
$|g_1| \leq f_1 \quad |g_2| \leq f_2$

$\therefore |L(g)| \leq |L(g_1)| + |L(g_2)| \leq \lambda(f_1) + \lambda(f_2)$

Taking sup ovell all such g:

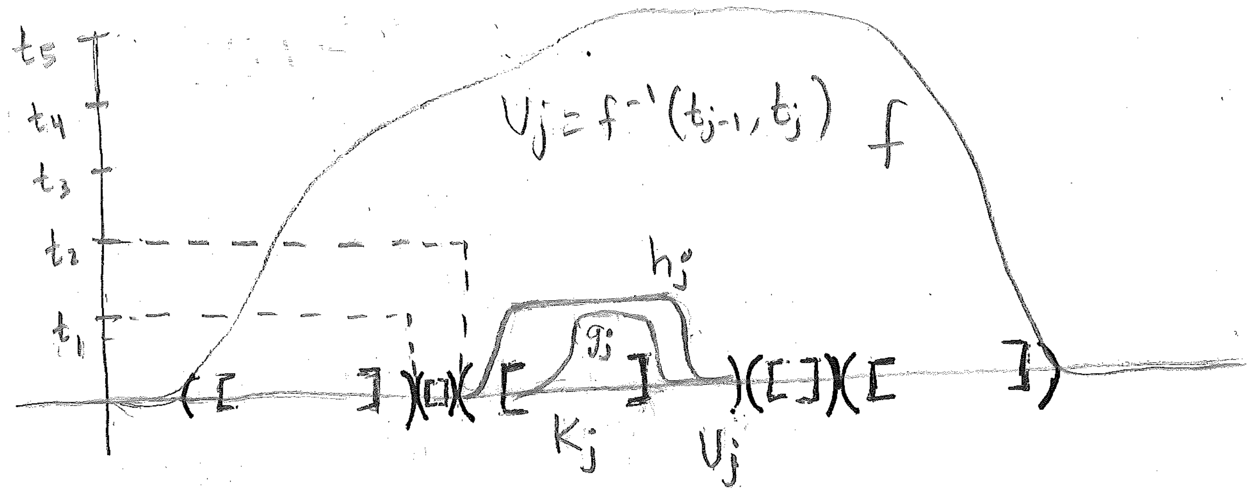
$\lambda(f_1+f_2) \leq \lambda(f_1) + \lambda(f_2)$

Step 6:

Claim: $\lambda(f) = \int_{\mathbb{R}^n} f \, d\mu \quad \forall f \in C_c^+(\mathbb{R}^n)$.

Let $\varepsilon > 0$. Choose
$0 = t_0 < t_1 < t_2 < \cdots < t_N = 2\|f\|_\infty$
$0 < t_j - t_{j-1} < \varepsilon$
$\mu(f^{-1}(t_j)) = 0 \quad \forall j = 1, \dots, N$
$U_j = f^{-1}(t_{j-1}, t_j)$

Let $K_j \subset U_j$, $\mu(U_j \setminus K_j) < \frac{\varepsilon}{N}$



$$\exists\, g_j \in C_c(\mathbb{R}^n, \mathbb{R}^m),\ |g_j| \le 1,\ \operatorname{spt}(g_j) \subset U_j.$$

$$\mu(U_j) - \frac{\varepsilon}{N} \le |L(g_j)| \le \mu(U_j)$$

$$\exists\, h_j \in C_c^+(\mathbb{R}^n),\ \operatorname{spt}(h_j) \subset U_j,\ 0 \le h_j \le 1$$

$$h_j \equiv 1 \text{ on } \underline{K_j \cup \operatorname{spt}(g_j)}$$

<u>Claim</u> $\mu(U_j) - \frac{\varepsilon}{N} \le \lambda(h_j) \le \mu(U_j)$

$$\begin{aligned} \lambda(h_j) &\ge |L(g_j)| \quad ; \quad g_j \le h_j \\ &\ge \mu(U_j) - \frac{\varepsilon}{N} \end{aligned}$$

$$\begin{aligned} \lambda(h_j) &= \sup\{|L(g)| : g \in C_c(\mathbb{R}^n, \mathbb{R}^m),\ |g| \le h_j\} \\ &\le \sup\{|L(g)| : g \in C_c(\mathbb{R}^n; \mathbb{R}^m),\ |g| \le 1,\ \operatorname{spt}(g) \subset U_j\} \\ &\le \mu(U_j). \end{aligned}$$

Recalling the Definition of integral: 8.90

$$\underline{\sum_{j=1}^{N} t_{j-1}\,\mu(U_j)} \le \int_{\mathbb{R}^n} f\,d\mu \le \underline{\sum_{j=1}^{N} t_j\,\mu(U_j)} \qquad (1)$$

Claim 1:

$$\underline{\sum_{j=1}^{N} t_{j-1}\,\mu(U_j) - \varepsilon t_N} \le \lambda(f) \le \varepsilon \|f\|_{L^\infty} + \underline{\sum_{j=1}^{N} t_j\,\mu(U_j)} \qquad (2)$$

From (1) + (2):

$$\left|\lambda(f) - \int_{\mathbb{R}^n} f\,d\mu\right| \le \sum_{j=1}^{N} (t_j - t_{j-1})\mu(U_j) + \varepsilon\|f\|_{L^\infty} + \varepsilon t_N$$

$$\le \varepsilon \sum_{j=1}^{N} \mu(U_j) + \varepsilon\|f\|_{L^\infty} + \varepsilon t_N$$

$$\le \varepsilon\mu(\operatorname{spt}(f)) + \varepsilon\|f\|_{L^\infty} + 2\|f\|_{L^\infty}\varepsilon$$

$$= \varepsilon\,\mu(\operatorname{spt}(f)) + 3\varepsilon\|f\|_{L^\infty}$$

$\varepsilon \to 0$ gives

$$\left|\lambda(f) - \int_{\mathbb{R}^n} f\,d\mu\right| = 0$$

$$\therefore \lambda(f) = \int_{\mathbb{R}^n} f\,d\mu.$$

Proof of Claim 1 :

$$f \geq \sum_{j=1}^{N} f h_j \implies$$

$$\implies \lambda(f) \geq \lambda\left(\sum_{j=1}^{N} f h_j\right)$$

$$= \sum_{j=1}^{N} \lambda(f h_j)$$

$$\geq \sum_{j=1}^{N} \lambda(t_{j-1} h_j) \; ; \quad \text{since } f h_j \geq t_{j-1} h_j \text{ on } U_j$$

$$= \sum_{j=1}^{N} t_{j-1} \lambda(h_j)$$

$$\geq \sum_{j=1}^{N} t_{j-1} \left(\mu(U_j) - \frac{\varepsilon}{N}\right)$$

$$= \sum_{j=1}^{N} t_{j-1} \mu(U_j) - \sum_{j=1}^{N} \frac{\varepsilon}{N} t_{j-1}$$

$$\geq \sum_{j=1}^{N} t_{j-1} \mu(U_j) - \sum_{j=1}^{N} \frac{\varepsilon}{N} t_N$$

$$= \sum_{j=1}^{N} t_{j-1} \mu(U_j) - \frac{\varepsilon}{\cancel{N}} \cdot \cancel{N} t_N$$

$$= \sum_{j=1}^{N} t_{j-1} \mu(U_j) - t_N \varepsilon$$

$$\therefore \quad \lambda(f) \geq \sum_{j=1}^{N} t_{j-1} \mu(U_j) - \varepsilon t_N$$

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$$\lambda(f) = \lambda\left(f - f\sum_{j=1}^{N} h_j\right) + \lambda\left(f\sum_{j=1}^{N} h_j\right)$$

linearity

$$\leq \operatorname{Sup}\left\{|L(g)| : |g| \leq f - f\sum_{j=1}^{N} h_j\right\} + \sum_{j=1}^{N} \lambda(f h_j)$$

$f h_j \leq t_j h_K$ in U_j & $\lambda(h_j) \leq \mu(U_j)$

$$\leq \operatorname{Sup}\left\{|L(g)| : |g| \leq \|f\|_{L^\infty} \chi_A\right\} + \sum_{j=1}^{N} t_j \mu(U_j)$$

$A := \{x : f(1 - \sum_{j=1}^{N} h_j) > 0\}$, A is open

$$= \|f\|_{L^\infty} \operatorname{Sup}\left\{|L(g)| : |g| \leq \chi_A\right\} + \sum_{j=1}^{N} t_j \mu(U_j)$$

Since $\lambda(cf) = c\lambda f$

$$= \|f\|_{L^\infty} \mu(A) + \sum_{j=1}^{N} t_j \mu(U_j)$$

$\Downarrow$

Since $\{g : |g| \leq \chi_A\} = \{g : \operatorname{spt}(g) \subset A, |g| \leq 1\}$

$$= \|f\|_{L^\infty} \mu\left(\bigcup_{j=1}^{N} (U_j - \{h_j = 1\})\right) + \sum_{j=1}^{N} t_j \mu(U_j)$$

$$\leq \|f\|_{L^\infty} \sum_{j=1}^{N} \mu(U_j - K_j) + \sum_{j=1}^{N} t_j \mu(U_j)$$

$$\leq \varepsilon \|f\|_{L^\infty} + \sum_{j=1}^{N} t_j \mu(U_j)$$