We have proven the following Riesz Representation Theorem:

<u>RRT3</u>: Let $L : C_c(\mathbb{R}^n; \mathbb{R}^m) \to \mathbb{R}$ be a linear functional satisfying:

$$\sup\{L(f) \mid f \in C_c(\mathbb{R}^n; \mathbb{R}^m),\ |f| \le 1,\ \operatorname{spt}(f) \subset K\} < \infty$$

for each compact set $K \subset \mathbb{R}^n$. Then:
$\exists\ \mu$ Radon measure on $\mathbb{R}^n$ and a μ-measurable function $\sigma : \mathbb{R}^n \to \mathbb{R}^m$ such that:

(i) $|\sigma(x)| = 1$ for μ-a.e. x, and

(ii) $L(f) = \int_{\mathbb{R}^n} f \cdot \sigma \, d\mu, \quad \forall f \in C_c(\mathbb{R}^n; \mathbb{R}^m)$

<u>Corollary of RRT3</u>:

Assume $L : C_c^\infty(\mathbb{R}^n) \to \mathbb{R}$ is linear and nonnegative, so that:

(*) $L(f) \geq 0$ for all $f \in C_c^\infty(\mathbb{R}^n)$, $f \geq 0$.

Then there exists a Radon measure μ on $\mathbb{R}^n$ such that:

(**) $L(f) = \int_{\mathbb{R}^n} f \, d\mu$ for all $f \in C_c^\infty(\mathbb{R}^n)$.

Proof of Corollary:

Fix $K \subset \mathbb{R}^n$ compact set.

Let $\varphi \in C_c^\infty(\mathbb{R}^n)$, $0 \le \varphi \le 1$, $\varphi \equiv 1$ on K.

Let $f \in C_c^\infty(\mathbb{R}^n)$, $\operatorname{spt}(f) \subset K$. Define:

$$g := \|f\|_\infty \varphi - f.$$

Note that $g \geq 0$, $g \in C_c^\infty(\mathbb{R}^n)$

$(*) \Rightarrow L(g) \geq 0$

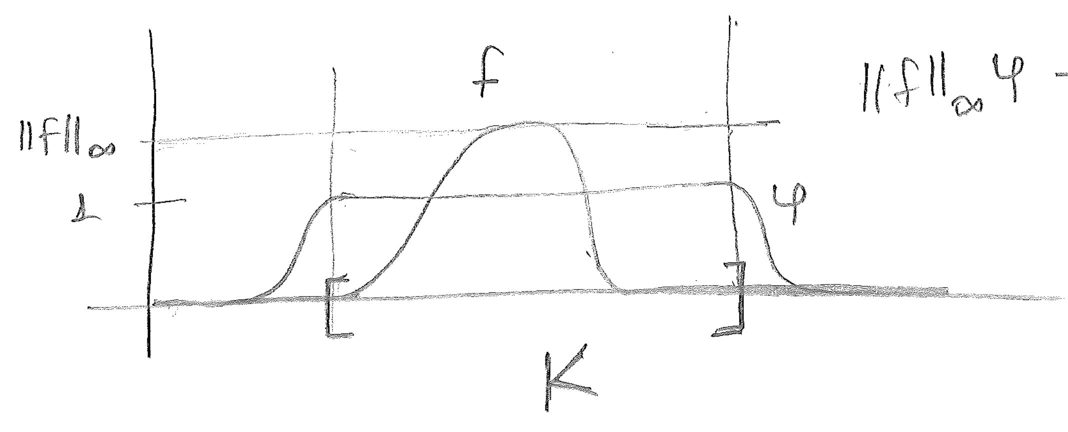
$\therefore L(\|f\|_\infty \varphi - f) \geq 0$

$\therefore \|f\|_\infty L(\varphi) - L(f) \geq 0$

$\Rightarrow L(f) \le L(\varphi) \|f\|_\infty$

$\therefore$ (**A**) $\boxed{L(f) \le C(K) \|f\|_\infty}$, $\forall f \in C_c^\infty(\mathbb{R}^n)$ $\operatorname{spt}(f) \subset K$.

with $C = L(\varphi)$. Note that C depends on K



$\|f\|_\infty \varphi - f \geq 0$

Let $f \in C_c^\infty(\mathbb{R}^n)$, $\operatorname{spt}(f) \subset K$.
define $\tilde{f} := -f$, Then:

$$\tilde{f} \in C_c^\infty(\mathbb{R}^n), \quad \operatorname{spt}(\tilde{f}) \subset K.$$

By (A):

$$L(f) \leq C\|f\|_\infty, \quad \text{and}$$
$$L(\tilde{f}) \leq C\|\tilde{f}\|_\infty.$$

$\therefore$

$$L(f) \leq C\|f\|_\infty \quad \text{and} \quad L(-f) \leq C\|-f\|_\infty = C\|f\|_\infty$$

$\therefore$ $$L(f) \leq C\|f\|_\infty, \text{ and } -L(f) \leq C\|f\|_\infty$$

$\therefore$ (B) $$\boxed{|L(f)| \leq C(K)\|f\|_\infty}, \quad \forall f \in C_c^\infty(\mathbb{R}^n), \ \operatorname{spt}(f) \subset K.$$

Recall:

Thm: Let E be a dense subset of a metric space X, and let f be a uniformly continuous real function defined on E. Then f has a continuous extension from E to X. Moreover, the extension is unique.

Recall: Hahn - Banach:

Thm: X normed linear space

$Y \subset X$ subspace.

$f : Y \to \mathbb{R}$,

$|f(x)| \leq M \|x\| \quad \forall x \in Y$

Then;

$\exists \quad g : X \to \mathbb{R}$ linear,

$g = f$ on Y, and:

$|g(x)| \leq M \|x\|, \quad \forall x \in X$.

Remark: $C_c^\infty(\mathbb{R}^n)$ is dence in $C_c(\mathbb{R}^n)$.

Let $f \in C_c(\mathbb{R}^n)$, $spt(f) \subset K$. We consider the convolutions

$f_\varepsilon = f * \rho_\varepsilon$, $spt(f_\varepsilon) \subset K$ for ε small.

Then

$f_\varepsilon \to f$ uniformly. on K

$\therefore \|f_\varepsilon - f\|_\infty \to 0$.

(We consider $C_c^\infty(\mathbb{R}^n)$ and $C_c(\mathbb{R}^n)$ as Normed linear spaces with the sup-norm).

We can not use previous extension theorems in this proof. Instead, from (B) and Remark we extend L to a linear mapping:

$$\tilde{L} : C_c(\mathbb{R}^n) \to \mathbb{R}$$

as follows:

Given $f \in C_c(\mathbb{R}^n)$, we define:

$$\tilde{L}(f) = \lim_{\varepsilon \to 0} L(f_\varepsilon), \quad f_\varepsilon = f * \rho_\varepsilon .$$

Give any $\varepsilon_j \to 0$ we have

$$|L(f_{\varepsilon_j}) - L(f_{\varepsilon_i})| = |L(f_{\varepsilon_j} - f_{\varepsilon_i})|$$
$$\leq C(\mathrm{spt}(f)) \, \| f_{\varepsilon_j} - f_{\varepsilon_i} \|_\infty .$$

Since $f_\varepsilon \to f$ uniformly, then $\{f_{\varepsilon_j}\}$ is Cauchy in the sup norm.

$\therefore$ $\{L(f_{\varepsilon_j})\}$ is Cauchy in $\mathbb{R}$, and hence it converges to a number, denoted by $\tilde{L}(f)$.

$\therefore$ $\tilde{L}$ is well defined.

Clearly, $\tilde{L}$ is linear.

<u>Claim</u>: $\tilde{L}$ satisfies the hypothesis of RRT3.

Fix $K \subset \mathbb{R}^n$ compact set. and $f \in C_c(\mathbb{R}^n)$, $spt(f) \subset K$, $|f| \leq 1$.

Then $$\tilde{L}(f) = \lim_{\varepsilon \to 0} L(f_\varepsilon)$$
$$\leq \lim_{\varepsilon \to 0} C(K) \|f_\varepsilon\|_\infty \rightarrow (C)$$

From $$\|f_\varepsilon - f\|_\infty \to 0$$

we have
$$|(f_\varepsilon - f)(x)| \leq 1 \quad \forall x, \ \forall \varepsilon \leq \varepsilon_0$$

$$\therefore \ |f_\varepsilon(x)| \leq 1 + |f(x)|$$
$$\leq 2 \quad \forall x \in \mathbb{R}^n, \ \forall \varepsilon \leq \varepsilon_0$$

$$\therefore \ \boxed{\tilde{L}(f) \leq 2C(K) < \infty \ \forall f \in C_c(\mathbb{R}^n), |f| \leq 1, \ spt(f) \subset K.}$$

From RRT3, it follows that $\exists\, \mu, \sigma$ s.t.

$$\tilde{L}(f) = \int_{\mathbb{R}^n} f \sigma \, d\mu \quad \forall f \in C_c(\mathbb{R}^n)$$

In particular:

$$L(f) = \int_{\mathbb{R}^n} f \sigma \, d\mu \quad \forall f \in C_c^\infty(\mathbb{R}^n)$$

RRT3 says $\sigma = \pm 1$ μ-a.e. Since:

$$L(f) \geq 0 \quad \forall f \in C_c^\infty(\mathbb{R}^n),\ f \geq 0,$$

we conclude $\sigma = 1$ μ-a.e.

That is:

$$\boxed{L(f) = \int_{\mathbb{R}^n} f \, d\mu, \quad \forall f \in C_c^\infty(\mathbb{R}^n)}$$

where μ a Radon measure in $\mathbb{R}^n$