We have already introduced two important partial differential equations:

1.- Laplace's equation:

$$\Delta u = \sum_{i=1}^{n} u_{x_i x_i} = 0.$$

2.- Heat equation:

$$u_t - \Delta u = 0$$

We want to introduce now two more equations:

3.- Wave equation

$$u_{tt} - \Delta u = 0$$

4.- Scalar Conservation law

$$u_t + \operatorname{div} \vec{F}(u) = 0$$

and systems of Conservation laws.

We have derived in an earlier lesson the heat equation. We want to derive now the wave equation from the Maxwell's equations:

$$(ME)\begin{cases} \vec{E}_t = \nabla\times\vec{B} - \vec{J} & \text{(Ampere's law)} \\ \vec{B}_t = -\nabla\times\vec{E} & \text{(Faraday's law)} \\ \operatorname{div}\vec{B} = 0 & \text{(No negative sources)} \\ \operatorname{div}\vec{E} = \rho & \text{(Gauss' law)} \end{cases}$$

$\rho(t,x,y,z)$: Charge density $\qquad \vec{E}$: Electric Field

$\vec{J}(t,x,y,z)$: Current density $\qquad \vec{B}$: Magnetic Field.

$$\operatorname{div}\vec{B} = \vec{0} \Rightarrow \boxed{\vec{B} = \nabla\times\vec{F}}, \text{ some } \vec{F}$$

$$\vec{0} = \vec{B}_t + \nabla\times\vec{E} = \frac{\partial}{\partial t}(\nabla\times\vec{F}) + \nabla\times\vec{E}$$

$$= \nabla\times\vec{E} + \nabla\times\frac{\partial\vec{F}}{\partial t}$$

$$= \nabla\times\left(\vec{E} + \frac{\partial\vec{F}}{\partial t}\right) \Rightarrow \exists\Phi \text{ s.t.}$$

$$\therefore \boxed{\vec{E} + \frac{\partial\vec{F}}{\partial t} = -\nabla\Phi} \quad (1)$$

$$\rho = \operatorname{div}\vec{E} = \operatorname{div}\left(-\nabla\Phi - \frac{\partial\vec{F}}{\partial t}\right)$$

$$= -\operatorname{div}(\nabla\Phi) - \frac{\partial}{\partial t}(\operatorname{div}\vec{F})$$

$$\Rightarrow \rho = -\Delta\Phi - \frac{\partial}{\partial t}(\operatorname{div}\vec{F})$$

$$\therefore \boxed{\frac{\partial}{\partial t}(\operatorname{div}(\vec{F}) + \Delta\Phi = -\rho} \rightarrow (2)$$

$$\circ \; -\vec{J} = \vec{E}_t - \nabla\times\vec{B}$$

$$= \frac{\partial}{\partial t}\left(-\nabla\Phi - \frac{\partial\vec{F}}{\partial t}\right) - \underline{\underline{\nabla\times(\nabla\times\vec{F})}}$$

$$= -\frac{\partial}{\partial t}\nabla\Phi - \frac{\partial^2\vec{F}}{\partial t^2} - \underline{\underline{\nabla(\nabla\cdot\vec{F}) + \Delta\vec{F}}}$$

$$\therefore \frac{\partial^2\vec{E}}{\partial t^2} - \Delta\vec{F} = -\frac{\partial}{\partial t}\nabla\Phi - \nabla(\nabla\cdot\vec{F}) + \vec{J}$$

$$\therefore \boxed{\frac{\partial^2\vec{F}}{\partial t^2} - \Delta\vec{F} = -\nabla\left(\frac{\partial\phi}{\partial t} + \nabla\cdot\vec{F}\right) + \vec{J}} \rightarrow (3)$$

We have freedom in choosing $\vec{F}$. We can use $\vec{F}+\nabla f$, (any f), instead of $\vec{F}$. We can use:

$$\vec{B} = \nabla\times(\vec{F}+\nabla f),$$

because $\nabla\times(\nabla f) = \vec{0}$.

(1) is still true for:

$$\vec{F} + \nabla f \quad \text{and} \quad \Phi - \frac{\partial f}{\partial t}$$

instead of

$$\vec{F} \quad \text{and} \quad \Phi$$

Indeed:

$$\vec{E} + \frac{\partial}{\partial t}(\vec{F} + \nabla f) = \underbrace{\vec{E} + \frac{\partial}{\partial t}\vec{F}}_{=} + \frac{\partial}{\partial t}\nabla f$$

$$= -\nabla\Phi + \frac{\partial}{\partial t}\nabla f$$

$$= -\nabla\left(\Phi - \frac{\partial f}{\partial t}\right)$$

$$\therefore \quad \boxed{\vec{E} + \frac{\partial}{\partial t}(\vec{F} + \nabla f) = -\nabla\left(\Phi - \frac{\partial f}{\partial t}\right)} \quad (1)'$$

Define:

$$\vec{G} = \vec{F} + \nabla f$$

$$\Upsilon = \Phi - \frac{\partial f}{\partial t}$$

With this choice,
equations (2) and (3) are:

$$\boxed{\begin{aligned} &(2)' \quad \Delta\Psi + \frac{\partial}{\partial t}(\operatorname{div}\vec{G}) = -\rho \\ &(3)' \quad \frac{\partial^2\vec{G}}{\partial t^2} - \Delta\vec{G} = -\nabla\left(\frac{\partial\Psi}{\partial t} + \nabla\cdot\vec{G}\right) + \vec{J} \end{aligned}}$$

We want to choose f so that:

$$\frac{\partial\Psi}{\partial t} + \nabla\cdot\vec{G} = 0$$

$$\therefore \quad \frac{\partial}{\partial t}\left(\Phi - \frac{\partial f}{\partial t}\right) + \nabla\cdot(\vec{F} + \nabla f) = 0$$

$$\therefore \quad \frac{\partial\Phi}{\partial t} - \frac{\partial^2 f}{\partial t^2} + \nabla\cdot\vec{F} + \Delta f = 0$$

$$\therefore \quad \Delta f - \frac{\partial^2 f}{\partial t^2} = -\underbrace{\left(\operatorname{div}\vec{F} + \frac{\partial\Phi}{\partial t}\right)}_{\text{A Known function}}$$

Solving this equation for f we have, with such f, that:

$$\frac{\partial \psi}{\partial t} + \nabla \cdot \vec{G} = 0.$$

Thus, (2)' + (3') reduce to:

$$\Delta \psi + \frac{\partial}{\partial t}\left(-\frac{\partial \psi}{\partial t}\right) = -\rho \ ; \text{ or.}$$

$$\boxed{\frac{\partial^2 \psi}{\partial t^2} - \Delta \psi = \rho}$$ Wave equation.

and:

$$\frac{\partial^2 \vec{G}}{\partial t^2} - \Delta \vec{G} = -\nabla \cancelto{0}{\left(\frac{\partial \psi}{\partial t} + \nabla \cdot \vec{G}\right)} + \vec{J}$$

or

$$\boxed{\frac{\partial^2 \vec{G}}{\partial t^2} - \Delta \vec{G} = \vec{J}}$$ Wave equation

9.116

To indicate the <u>wavelike nature</u> of solutions to the wave equation, note that for any function f,

$$\Phi(t, x, y, z) = f(x-t)$$

solves the wave equation:

$$\frac{\partial^2 \Phi}{\partial t^2} - \Delta \Phi = 0$$

This solution propagates the graph of f like a wave.

$\therefore$ Solutions to Maxwell equations are wavelike in nature.

Maxwell's great achievement! It soon led to Hertz's discovery of radio waves!!

See Chapter 7 of PDE book by Evans for the analysis of the Heat and wave equations using <u>weak convergence</u> arguments!

Indeed.

Consider the PDE's:

$$(1)\begin{cases} u_t - \Delta u = f & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0,T] \\ u = g & \text{on } U \times \{t=0\} \end{cases}$$

$$U_T = U \times [0,T]$$

Theorem 7 (PDE Book).

Assume
$g \in C^\infty(\bar{U})$, $f \in C^\infty(\bar{U}_T)$,

and the m^{th} - order compatibility conditions hold for $m = 0, 1, \ldots$. Then the Parabolic initial - boundary problem has a unique solution

$$u \in C^\infty(\bar{U}_T).$$

Consider now:

$$
② \begin{cases} u_{tt} + \Delta u = f & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0,T] \\ u = g, \quad u_t = h & \text{on } U \times \{t=0\}. \end{cases}
$$

<u>Theorem 7</u> (PDE Book)

Assume $g, h \in C^\infty(\bar{U})$, $f \in C^\infty(\bar{U}_T)$, and the m^{th}- order compatibility conditions hold for $m = 0, 1, \ldots$.

Then the hyperbolic initial/boundary-value problem ② has a unique solution

$$u \in C^\infty(\bar{U}_T).$$

The proof of previous Theorem in Chapter 7 (PDE book) illustrate the general method used in PDE for many equations:

$\boxed{\text{step } 1}$: Prove the existence of a "weak solution" of the equation in some space of functions. This "weak solution" $\underline{u}$ is often the "weak limit" of a sequence of solutions $\underline{u_K}$ to an approximate equation.

In many cases, a compactness theorem is applied to $\{u_K\}$, given that one could have:

$$\|u_K\| \leq M \quad \forall K = 1, 2, \ldots$$

For the heat and wave equations in Theorem 7, the compactness is given by Theorem 291.2 proved in this class: A sequence

$\{u_K\} \subset L^2$ with

$$\|u_K\| \leq M \quad K=1,2,\ldots$$

has a subsequence $\{u_{K_j}\}$ that converges weakly to some $u \in L^2$.

$\boxed{\text{Step 2}}$: Having now the "weak solution u", then we prove <u>regularity</u> for u. This process consists in proving properties of u. Originally u belongs to a particular "<u>space of functions</u>"; for exemple L^2. We now ask the questions: Is u continuous? If u differentiable in the classical sense? What is the behavior of u at infinity? Answering these type of questions means "to study regularity".

Systems of Conservation laws.

We want to investigate a vector function

$$\vec{u} = \vec{u}(t,x) = (u^1(x,t), \dots, u^m(x,t)),$$

$$x \in \mathbb{R}^n, \quad t > 0$$

The components are the densities of various conserved quantities.

The:

$$\int_U \vec{u}(x,t)\, dx$$

represents the total amount of these quantities within U at time t.

Physical truth: "The rate of change within U is governed by a flux function $\vec{F} : \mathbb{R}^m \to M^{m\times n}$, which control the rate of loss or increase of $\vec{u}$ through ∂U".

We say this in mathematical language as follows:

$$\frac{d}{dt}\int_U \vec{u}(t,x)\,dx = -\int_{\partial U} \vec{F}(\vec{u})\cdot\vec{\nu}\,ds$$

$$\therefore \int_U \vec{u}_t\,dx = -\int_{\partial U} \vec{F}(\vec{u})\cdot\vec{\nu}\,ds$$ $\vec{\nu}$ outward unit normal

$$= -\int_U \operatorname{div}\vec{F}(\vec{u})\,dx$$, Divergence theorem.

$$\therefore \int_U \vec{u}_t + \operatorname{div}\vec{F}(\vec{u}) = 0. \qquad (1)$$

Since (1) is true for any open set U it follows that

$$\vec{u}_t + \operatorname{div}\vec{F}(\vec{u}) = 0$$

We have:

General system of Conservation laws:

$$(\text{GSCL})\begin{cases} \vec{u}_t + \operatorname{div}\vec{F}(\vec{u}) = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

$g = (g^1,\dots,g^m)$ is the initial distribution of $\vec{u} = (u^1,\dots,u^m)$.

Current State of Research for GSCL:

At present, a good mathematical understanding of GSCL is largely unavailable.

Significant theories have been obtained in the following cases.

1.- $m=1$, $n\geq 1$ (that is, one equation but many dimensions):

$$u_t + \mathrm{div}_x f(u) = 0, \qquad \begin{array}{l} u : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R} \\ f : \mathbb{R} \to \mathbb{R}^n \\ f(x) = (f^1(x), \dots, f^n(x)) \end{array}$$

2.- $m\geq 1$, $n=1$ (that is, many equations but dimension 1):

$$\begin{array}{l} u^1_t + (f^1(\vec{u}))_x = 0 \\ u^2_t + (f^2(\vec{u}))_x = 0 \\ \vdots \\ u^m_t + (f^m(\vec{u}))_x = 0 \end{array}, \qquad \begin{array}{l} \vec{u} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^m \\ \vec{u} = (u^1, \dots, u^m) \\ f^i : \mathbb{R}^m \to \mathbb{R} \end{array}$$

Examples:

1.- $n=1$, $m=1$
Burger's equation

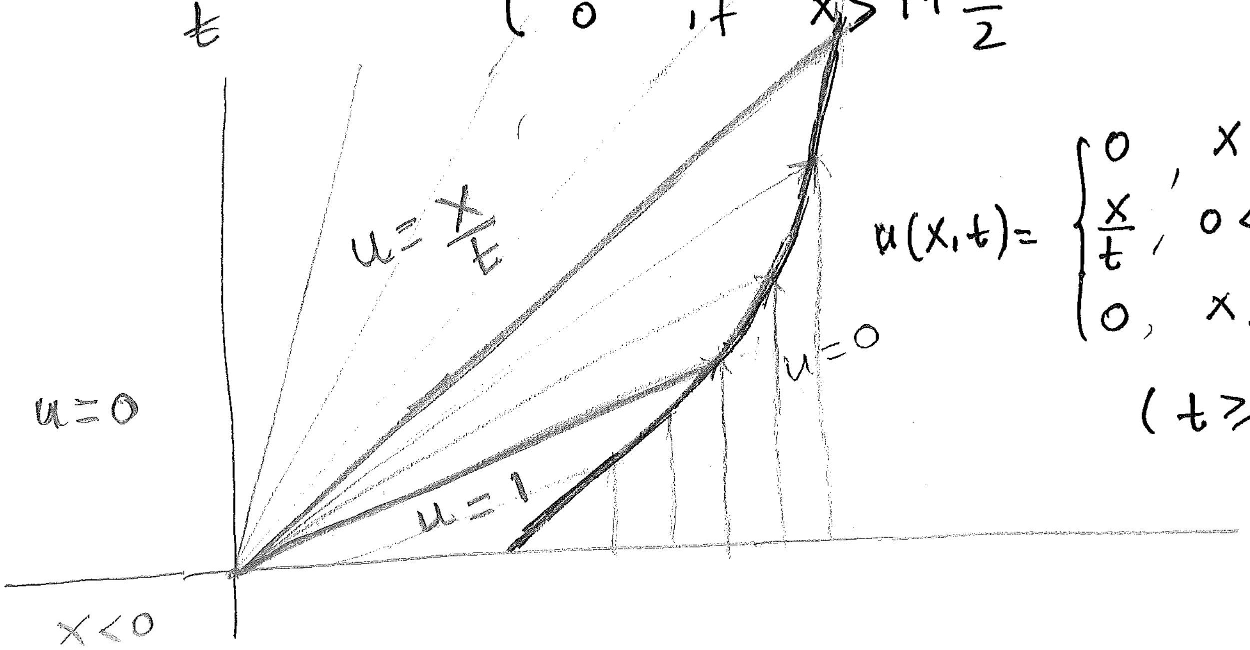
$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 & \text{in } \mathbb{R}\times(0,\infty) \\ u = g & \text{on } \mathbb{R}\times\{t=0\} \end{cases}$$

See Chapter 3 (PDE book):

If $g(x) = \begin{cases} 0 & \text{if } x<0 \\ 1 & \text{if } 0\le x\le 1 \\ 0 & \text{if } x>1 \end{cases}$

"Solution" is:

$$u(x,t) = \begin{cases} 0 & \text{if } x<0 \\ \frac{x}{t} & \text{if } 0<x<t \\ 1 & \text{if } t<x<1+\frac{t}{2} \\ 0 & \text{if } x>1+\frac{t}{2} \end{cases} \qquad (0\le t\le 2).$$



$$u(x,t) = \begin{cases} 0, & x<0 \\ \frac{x}{t}, & 0<x<(2t)^{1/2} \\ 0, & x>(2t)^{1/2} \end{cases}$$

$(t\ge 2)$.

Ex: Euler's equations for compressible gas flow in one dimension $(m=3, \; n=1)$.

$$\rho_t + (\rho v)_x = 0 \qquad \text{(Conservation of mass).}$$

$$(\rho v)_t + (\rho v^2 + p)_x = 0 \qquad \text{(Conservation of momentum)}$$

$$(\rho E)_t + (\rho E v + p v)_x = 0 \qquad \text{(Conservation of energy).}$$

$p(\rho)$, pressure depends on density

ρ mass density.

v velocity.

E energy density per unit mass.

$$\vec{u} = (u^1, u^2, u^3) = (\rho, \rho v, \rho E)$$

$$f^1(z_1, z_2, z_3) = z_2$$

$$f^2(z_1, z_2, z_3) = \frac{(z_2)^2}{z_1} + p(z_1)$$

$$f^2(z_1, z_2, z_3) = \frac{z_2 z_3}{z_1} + p(z_1)\frac{z_2}{z_1}$$

Note: Not available theory for $n > 1$!!! In which space of functions shall we look for solutions?