- Consider the initial-value problem for multidimensional scalar conservation law ($m=1$, $n\geq 1$):

$$(SCL)\begin{cases} u_t + \mathrm{div}_x f(u) = 0 & \text{in } \mathbb{R}^n\times(0,\infty) \\ u = g & \text{on } \mathbb{R}^n\times\{t=0\}. \end{cases}$$

$$u: \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$$
$$f: \mathbb{R}\to\mathbb{R}^n, \quad f = (f^1, f^2, \dots, f^n).$$

Definition: The smooth functions

$$\eta: \mathbb{R}\to\mathbb{R}$$
$$q: \mathbb{R}\to\mathbb{R}^n$$

are an entropy / entropy-flux pair for (SCL) if

η is convex.

$\eta'(z) f'(z) = q'(z)$, $z\in\mathbb{R}$

We say that $u: \mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ satisfies the entropy condition if:

$$(EC)\begin{cases} \int_0^\infty \int_{\mathbb{R}^n} \eta(u)\varphi_t + q(u)\cdot\nabla\varphi \, dx\,dt \geq 0, \\ \text{for each } \varphi\in C_c^\infty(\mathbb{R}^n\times(0,\infty)),\ \varphi\geq 0, \\ \text{and for every entropy / entropy-flux pair } \eta, q. \end{cases}$$

Def: We call $u \in L^\infty(\mathbb{R}^n \times (0,\infty))$ an entropy solution of the (SCL) if u satisfies the entropy Condition (EC) for each entropy/entropy-flux pair (η, q) and $u(\cdot, t) \to g$ in L^1 as $t \to 0$

See Chapter 11, PDE Book for a motivation for the Entropy Condition

<u>Remark</u>:
Taking $(z) = \pm z$, $q(z) = \pm f(z)$ we deduce:

$$\int_0^\infty \int_{\mathbb{R}^n} u \varphi_t + f(u) \cdot \nabla \varphi \, dx\, dt = 0,$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n \times (0,\infty))$.

<u>Thm</u>: There exists a unique entropy solution $u \in L^\infty(\mathbb{R}^n \times (0,\infty))$ of Equation (SCL) for bounded initial data.

This Theorem was proven in.

- Kruzkov, S.N: "First order quasilinear equations in several independent variables." Math USSR-Sbornik, 10, 217-243 (1970).

We have seen that:

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If $\underline{u}$ is a bounded entropy solution of:

$$\underline{u_t + \mathrm{div}_x f(u) = 0}$$

then u satisfies:

$$\underline{L^u_\eta(\varphi) \geq 0}, \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^n \times (0,\infty)),$$

$\underline{\varphi \geq 0}$; where

$$L^u_\eta(\varphi) := \int_0^\infty \int_{\mathbb{R}^n} \eta(u)\varphi_t + q(u).\nabla\varphi \, dx\, dt.$$

Then, using the Corollary of the Riesz Representation Theory (RRT3) proven in class it follows that there exists, for each pair (η, q), a Radon measure μ_η (positive); such that:

$$\boxed{L^u_\eta(\varphi) = \int_{\mathbb{R}^n \times (0,\infty)} \varphi \, d\mu_\eta}, \quad (1)$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n \times (0,\infty))$.

Therefore, from (1):

$$\boxed{\int_0^\infty \int_{\mathbb{R}^n} \eta(u)\varphi_t + q(u)\cdot\nabla\varphi \, dx\, dt = \int_{\mathbb{R}^n\times(0,\infty)} \varphi \, d\mu_\eta}, \quad (2)$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n \times (0,\infty))$.

Assume for the moment that u, η, and q are smooth (we know only that $u \in L^\infty$).

Then, from (2):

$$\int_0^\infty \int_{\mathbb{R}^n} (\eta(u), q(u)) \cdot (\varphi_t, \nabla\varphi)\, dx\, dt = \int_{\mathbb{R}^n\times(0,\infty)} \varphi \, d\mu_\eta \quad (3)$$

Using integration by parts, (3) becomes:

$$-\int_0^\infty \int_{\mathbb{R}^n} \varphi \, \mathrm{div}_{t,x}(\eta(u), q(u))\, dx\, dt = \int_{\mathbb{R}^n\times(0,\infty)} \varphi \, d\mu_\eta, \quad (4)$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n \times (0,\infty))$. Since (4) is true for any smooth function φ, we conclude that:

$$\boxed{\mathrm{div}_{t,x}(\eta(u), q(u)) = -\mu_\eta} \quad (5)$$

In reality we have that $u \in L^\infty$ and hence we can only say that:

$$F_{u,\eta} := (\eta(u), q(u)) \in L^\infty$$

Hence, from

$$\int_0^\infty \int_{\mathbb{R}^n} \eta(u)\varphi_t + q(u)\cdot\nabla\varphi \, dx\, dt = \int_{\mathbb{R}^n \times [0,\infty)} \varphi \, d\mu_\eta ,$$

for all $\varphi \in C_c^\infty$, we conclude that

$$\boxed{\mathrm{div}_{t,x} (\eta(u), q(u)) = -\mu_\eta} \quad (6).$$

in the sense of distributions. This will be made clear in Chapter 10 of your textbook, which is dedicated to an introduction to the Theory of distributions.

Using (6) we can study the regularity of the entropy solution u. This analysis can be found in the reference:

"Structure of entropy solutions for Multi-dimensional Scalar Conservation laws", by De Lellis, Otto, Westdickenberg. Arch. Rational Mech. Anal. 170 (2003), 137-184.

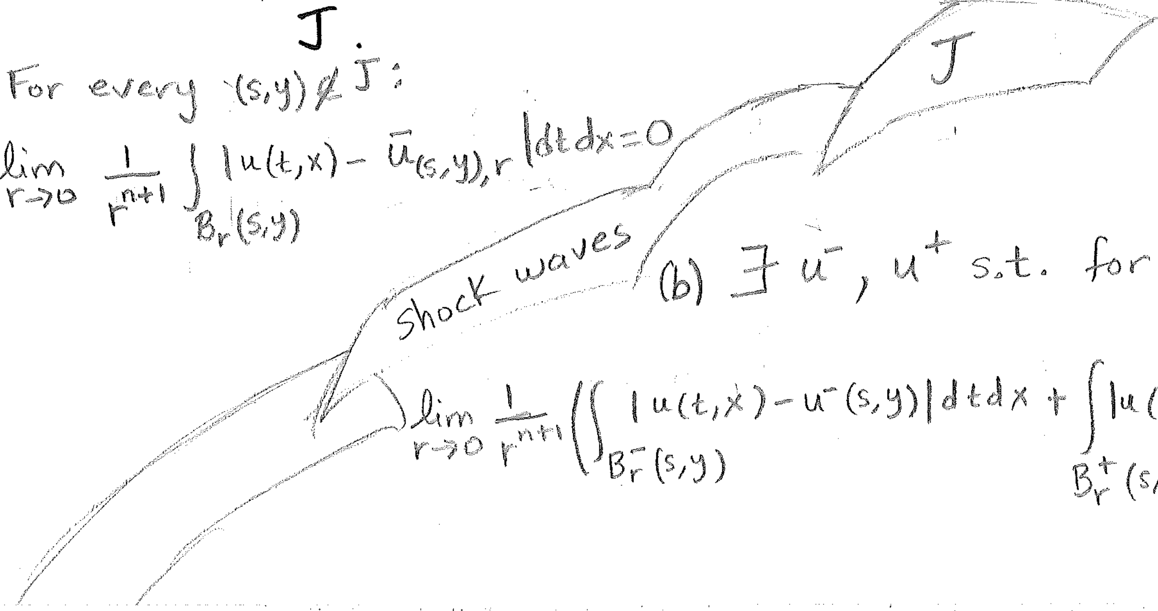
This regularity says:

<u>Thm</u>: There exists a rectifiable set J of dimension n such that

(a) u has vanishing mean oscillation at every $y \notin J$.

(b) u has left and right traces on J.

$J \subset \bigcup_{i=1}^{\infty} G_i$, G_i is a Lipschitz graph.



(a) For every $(s,y) \notin J$:

$$\lim_{r\to 0} \frac{1}{r^{n+1}} \int_{B_r(s,y)} |u(t,x) - \bar{u}_{(s,y),r}|\, dt\, dx = 0$$

(b) $\exists\, u^-, u^+$ s.t. for $\mathcal{H}^n$-a.e $(s,y) \in J$:

$$\lim_{r\to 0} \frac{1}{r^{n+1}} \left(\int_{B_r^-(s,y)} |u(t,x) - u^-(s,y)|\, dt\, dx + \int_{B_r^+(s,y)} |u(t,x) - u^+(s,y)|\, dt\, dx \right) = 0$$

We consider now the isentropic Euler equations in gas dynamics

$$(\text{IEE}) \begin{cases} \rho_t + (\rho v)_x = 0 \\ (\rho v)_t + (\rho v^2 + p(\rho))_x = 0 \end{cases}$$

$$\vec{u} = (u^1, u^2) = (\rho, \rho v)$$

Letting $m = \rho v$ we obtain:

$$(\text{IEE}) \begin{cases} \rho_t + m_x = 0 \\ m_t + \left(\frac{m^2}{\rho} + p(\rho)\right) = 0 \end{cases}$$

$$p(\rho) = \rho^{\gamma}.$$

It has been proven that there exists a unique entropy solution $\vec{u}(t,x) = (\rho(t,x), m(t,x))$ of (IEE) with

$$\rho(t,x),\ m(t,x) \in L^{\infty}(\mathbb{R} \times (0,\infty))$$

This existence theorem is based on Compactness Theorems that use the Compensated Compactness Theory.

Let $\vec{u} = (\rho, m)$ be the entropy solution of (IEE). This means; for every entropy pair (η, q):

$$\int_0^\infty \int_{\mathbb{R}} \eta(\vec{u})\varphi_t + q(\vec{u}) \cdot \varphi_x \, dx dt = \int_{\mathbb{R}} \varphi \, d\mu_\eta$$

for all $\varphi \in C_c^\infty(\mathbb{R} \times (0,\infty))$. In this case we have $\eta : \mathbb{R}^2 \to \mathbb{R}$ and $q : \mathbb{R}^2 \to \mathbb{R}$.

$$\therefore \quad \mathrm{div}_{t,x} (\eta(\vec{u}), q(\vec{u})) = -\mu_\eta$$

in the sense of distributions.

The next problem is to do regularity theory for $\vec{u}$: What can we say about the "shock waves" location? Is $\vec{u}$ continuous away the shock waves? Does $\vec{u}$ have traces on the shock waves? These are difficult questions. Based on current research, the answer to these questions might be advanced by solving first the following problem:

Research Question:

A Liouville-Type Theorem:

Let $\vec{u} = (\rho(t,x), m(t,x)) \in L^\infty(\mathbb{R}^2)$ be such that

$$\partial_t \eta(\vec{u}) + \partial_x q(\vec{u}) = 0$$

in the sense of distributions in $\mathbb{R}^2$, for every entropy pair (η, q). Then $\vec{u}$ must be constant.