(iv) Let $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$.

We know that:

$$f_\varepsilon = \varphi_\varepsilon * f \in C^\infty(\mathbb{R}^n).$$

We want to show now that:

$\varphi_\varepsilon * f \in L^p(\mathbb{R}^n)$ and $\|\varphi_\varepsilon * f\|_p \le \|f\|_p$.

Since $\varphi_\varepsilon \in L^1(\mathbb{R}^n)$ and $\|\varphi_\varepsilon\|_1 = 1$, the desired result follows from the following:

<u>Claim:</u> If $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$ and $g \in L^1(\mathbb{R}^n)$, then $f * g \in L^p(\mathbb{R}^n)$ and

$$\|f * g\|_p \le \|f\|_p \|g\|_1$$

Proof of the Claim:

Since $|f*g| \le |f| * |g|$ we need to prove the claim only for $\underline{f, g \ge 0}$.

<u>For $p = 1$:</u>

$$\int_{\mathbb{R}^n} (f*g)(x)\,dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(x-y) f(y)\,dy\,dx$$

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(x-y) f(y)\,dx\,dy$$

Here, we applied <u>Tonelli's Theorem</u> to interchange the order of integration. Recall that Tonelli's theorem hypothesis is only that the integrand is a non-negative function, and in this proof, $f, g \geq 0$.

$$\therefore \int_{\mathbb{R}^n} (f*g)(x)\,dx = \int_{\mathbb{R}^n} f(y)\,dy \int_{\mathbb{R}^n} g(x-y)\,dx$$

$$= \|f\|_1 \|g\|_1 .$$

<u>For $1 < p < \infty$:</u>

$$(f*g)(x) = \int_{\mathbb{R}^n} f(y)\,(g(x-y))^{1/p}\,(g(x-y))^{1-\frac{1}{p}}\,dy$$

$$\leq \left(\int_{\mathbb{R}^n} f^p(y)\,g(x-y)\,dy\right)^{1/p}\left(\int_{\mathbb{R}^n} g(x-y)^{p'-\frac{p'}{p}}\right)^{1/p'}$$

(Here we used Holder's inequality

$$= \left[(f^p * g)(x)\right]^{1/p}\left(\int_{\mathbb{R}^n} g(x-y)\,dy\right)^{1-\frac{1}{p}}$$

Since $\frac{1}{p}+\frac{1}{p'}=1 \Rightarrow \frac{p'}{p}+1=p'$

$$= \left[(f^p * g)(x)\right]^{1/p} \|g\|_1^{1-\frac{1}{p}}$$

$$\therefore \int_{\mathbb{R}^n} (f*g)^p(x) \leq \int_{\mathbb{R}^n} (f^p*g)(x)\,dx\, \|g\|_1^{p-1}$$

$$= \|f^p\|_1 \|g\|_1 \|g\|_1^{p-1} = \|f\|_p^p \|g\|_1^p \quad \blacksquare$$

We proceed to show:

$$\|f - f_\varepsilon\|_p \to 0.$$

Recall that:

<u>$C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$</u>

(Use Corollary 144.1 to show that step functions are dense in $L^p(\mathbb{R}^n)$ and then approximate step functions by continuous functions).

$\therefore \forall \eta > 0 \ \exists g \in C_c(\mathbb{R}^n)$ s.t:

$$\underline{\|f - g\|_p < \eta.}$$

We estimate:

$$\boxed{\|f - f_\varepsilon\|_p \leq \|f - g\|_p + \|g - g_\varepsilon\|_p + \|g_\varepsilon - f_\varepsilon\|_p} \quad (A)$$

g continuous $\Rightarrow$ $g_\varepsilon \to g$ uniformly on a compact set K with:

$$\text{Spt}(g_\varepsilon),\ \text{Spt}(g) \subset K.$$

$\therefore \exists \varepsilon_0$ s.t. $\quad |g_\varepsilon(x) - g(x)|^p < \dfrac{\eta^p}{\lambda(K)} \quad \forall x \in K, \forall \varepsilon \leq \varepsilon_0$

$\therefore$

$$\|g - g_\varepsilon\|_p = \left(\int_{\mathbb{R}^n} |g - g_\varepsilon|^p dx\right)^{1/p} = \left(\int_K |g(x) - g_\varepsilon(x)|^p\right)^{1/p}$$

$$\leq \left(\int_K \frac{\eta^p}{\lambda(K)} dx\right)^{1/p} = \left(\frac{\eta^p}{\lambda(K)} \lambda(K)\right)^{1/p}$$

$$= \eta \qquad , \quad \forall \varepsilon \leq \varepsilon_0$$

$\therefore$ $\underline{\|g - g_\varepsilon\|_p \leq \eta \qquad , \quad \forall \varepsilon \leq \varepsilon_0}$

We have also proved above that.

$$\|g_\varepsilon - f_\varepsilon\|_p = \|(g-f)_\varepsilon\|_p$$
$$\leq \|g - f\|_p$$
$$\leq \eta.$$

$\therefore$ $\underline{\|g_\varepsilon - f_\varepsilon\|_p \leq \eta.}$

From (A):

$$\|f - f_\varepsilon\|_p \leq \eta + \eta + \eta = 3\eta \qquad , \quad \forall \varepsilon \leq \varepsilon_0$$

this shows that:

$$\|f - f_\varepsilon\|_p \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Definition: Let $\Omega \subset \mathbb{R}^n$ be an open set. A linear functional T on $\mathcal{D}(\Omega)$ is a distribution if and only if for every compact set $K \subset \Omega$, there exist constants C and N such that

$$|T(\varphi)| \leq C \sum_{|\alpha| \leq N} \sup_{x \in K} |D^\alpha \varphi(x)|, \quad \forall \varphi \in \mathcal{D}(\Omega), \ spt(\varphi) \subset K.$$

If N can be chosen independent of the compact set K, and N is the smallest possible choice, the distribution is said to be of order N.

Define:

$$\|\varphi\|_{K;N} := \sum_{|\alpha| \leq N} \sup_{x \in K} |D^\alpha \varphi(x)|$$

Note:

$\|\varphi\|_{K;0}$ is the sup norm of φ on K.

Ex: Let μ be a signed Radon measure on Ω.

Define

$$T : \mathcal{D}(\Omega) \to \mathbb{R}$$

$$T(\varphi) = \int_\Omega \varphi(x)\, d\mu(x)$$

Fix $K \subset \Omega$ compact set, & let $\varphi \in \mathcal{D}(\Omega)$, $\mathrm{spt}(\varphi) \subset K$, then:

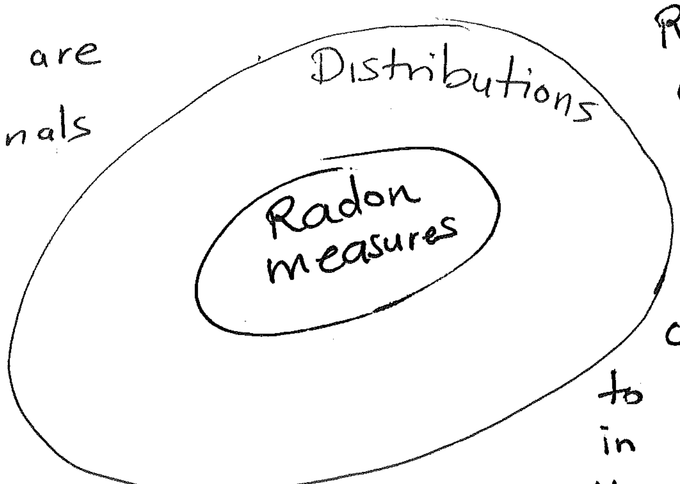
$$|T(\varphi)| \le \int_\Omega |\varphi|\, d|\mu| = \int_K |\varphi|\, d|\mu|$$

$$\le \|\varphi\|_{L^\infty} |\mu|(K)$$

$$= C\, \|\varphi\|_{K;0}, \quad C = |\mu|(K)$$

$\therefore$ "A Radon measure can be identified with a distribution of order 0".

Distributions are linear functionals defined on $C_c^\infty(\Omega)$ and continuous with respect to the topology in $C_c^\infty(\Omega)$ induced by the theory of locally convex topological vector spaces (l.c.t.v.s.).



Radon measures are linear functionals defined on $C_c(\Omega)$ and continuous with respect to the topology in $C_c(\Omega)$ induced by the theory of l.c.t.v.s.

Ex. Let $f \in L^1_{loc}(\Omega)$.

Define

$$T: \mathcal{D}(\Omega) \to \mathbb{R}$$
$$T(\varphi) = \int_\Omega \varphi(x) f(x)\, d\lambda(x).$$

Fix $K \subset \Omega$ compact set, a let $\varphi \in \mathcal{D}(\Omega)$, $\operatorname{Spt}(\varphi) \subset K$. Then:

$$\begin{aligned}
|T(\varphi)| &= \left| \int_\Omega \varphi(x) f(x)\, d\lambda(x) \right| \\
&\le \int_\Omega |\varphi|\, |f|\, d\lambda(x) = \int_K |\varphi|\, \underline{|f|\, d\lambda} \\
&\le \|\varphi\|_\infty \int_K |f|\, d\lambda \\
&= C\, \|\varphi\|_{K;0}, \qquad C = \int_K |f|\, d\lambda
\end{aligned}$$

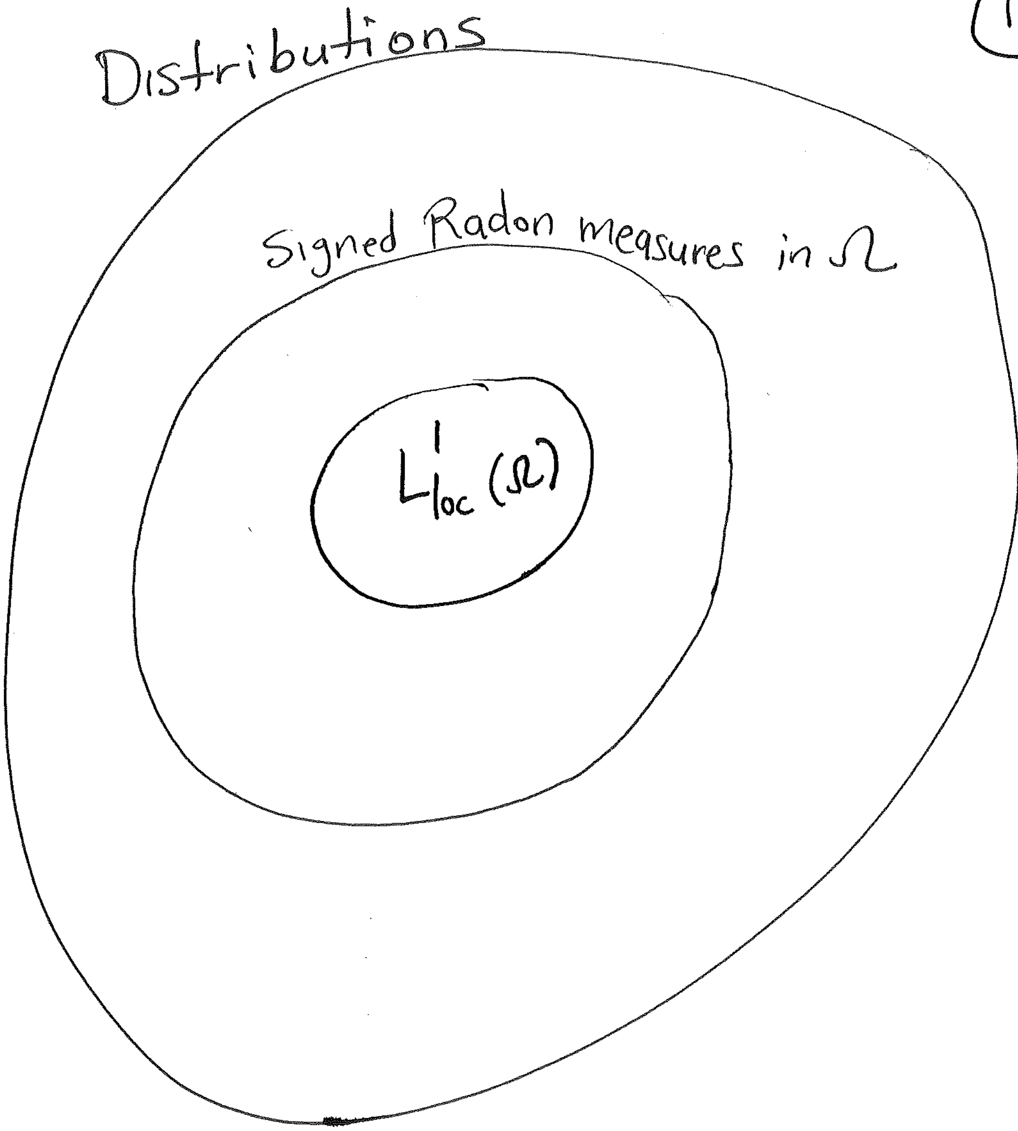
A locally integrable function can be identified with a Radon measure:

$$f \longrightarrow \mu := f\lambda,$$

and therefore identified with a distribution of order 0. In fact, given $f \in L^1_{loc}(\Omega)$ we associate the measure $f\lambda$ define as:

$$f\lambda(E) = \int_E f(x)\, d\lambda(x), \quad \forall E \in \mathcal{M}.$$

Then:



Ex: Consider the Dirac measure δ whose total mass is concentrated at the origin:

$$\delta(E)=\begin{cases}1 & 0\in E\\ 0 & 0\notin E.\end{cases}$$

The distribution identified with this measure is defined by:

$$T(\varphi)=\varphi(0), \quad \forall \varphi\in\mathcal{D}(\Omega)$$