Notes on the Total variation of a measure.

Def If μ is a measure, we define its total variation $|\mu|$ for every $E \in \mathcal{M}$ as follows:

$$|\mu|(E) := \sup\left\{ \sum_{i=0}^{\infty} |\mu(E_i)| : E_i \in \mathcal{M} \text{ are pairwise disjoint, } E \subset \bigcup_{i=1}^{\infty} E_i \right\}$$

Def: We also have that the measure $|\mu|$ is:

$$|\mu| = \mu^+ + \mu^-$$

Def: The norm of μ is $\|\mu\| := |\mu|(\mathbb{R}^n)$

Def: Let μ be a positive measure on the measure space $(\mathbb{R}^n, \mathcal{M})$ and let $f \in L^1(\mathbb{R}^n, \mu)$. Let $f\mu$ denote the measure given by:

$$f\mu(E) = \int_E f \, d\mu, \quad \forall E \in \mathcal{M}.$$

Lemma 1: Let $f\mu$ be the measure introduced in the previous definition. Then:

$$|f\mu|(E) = \int_E |f| \, d\mu, \quad \forall E \in \mathcal{M}$$

Proof of RRT3, Global version.

Thm: Let $C_0(\mathbb{R}^n)$ be the completion of the space $C_c(\mathbb{R}^n)$, which is endowed with the norm:

$$\|f\| = \sup\{|f(x)| : x \in \mathbb{R}^n\}.$$

Let $\mathcal{M}(\mathbb{R}^n) = \{\mu : \mu \text{ is a Borel regular signed measure}, |\mu|(\mathbb{R}^n) < \infty\}$.

$C_0(\mathbb{R}^n)$ is a Banach space with the norm:

$$\|f\| = \sup\{|f(x)| : x \in \mathbb{R}^n\}.$$

$C_0(\mathbb{R}^n)$ is the set of all functions $f : \mathbb{R}^n \to \mathbb{R}$ vanishing at infinity; i.e,

- $f \in C_0(\mathbb{R}^n)$ iff $\forall \varepsilon > 0$, $\exists K$ compact $|f(x)| \le \varepsilon$, $\forall x \in \mathbb{R}^n \setminus K$.

- Also, $f \in C_0(\mathbb{R}^n)$ iff. $\exists \{f_n\} \in C_c(\mathbb{R}^n)$ such that

$$f_n \to f \text{ uniformly}$$

or $\sup_{x \in \mathbb{R}^n} |f_n(x) - f(x)| \to 0$ as $n \to \infty$.

Then the map:

$$\Psi : \mathcal{M}(\mathbb{R}^n) \to (C_0(\mathbb{R}^n))^*$$

given by

$$\Psi(\mu)(f) = \int_{\mathbb{R}^n} f \, d\mu \quad , \quad f \in C_0(\mathbb{R}^n)$$

is an isometric isomorphism, that is, Ψ is one-to-one, on-to, and:

$$\underline{\|\mu\|} = |\mu|(\mathbb{R}^n) = \underline{\|\Psi(\mu)\|}$$

$$= \sup\{ |\Psi(\mu)(f)| : |f| \leq 1, \; f \in C_0(\mathbb{R}^n)\}$$

Proof:

Step 1: Ψ is well defined.

Clearly, $\Psi(\mu)$ is linear. Now:

$$|\Psi(\mu)(f)| = \left| \int_{\mathbb{R}^n} f \, d\mu \right|$$

$$\leq \int_{\mathbb{R}^n} |f| \, d|\mu|$$

$$\leq \|f\|_\infty \int_{\mathbb{R}^n} d|\mu|$$

$$= \|f\|_\infty |\mu|(\mathbb{R}^n)$$

$$= \|f\|_\infty \|\mu\|$$

$$\therefore |\Psi(\mu)(f)| \leq \|f\|_\infty \|\mu\|, \quad \forall f \in C_0(\mathbb{R}^n)$$

Therefore, $\underline{\Psi(\mu) \in (C_0(\mathbb{R}^n))^* \text{ and } \|\Psi(\mu)\| \leq \|\mu\|}$

Step 2: Ψ is on-to.

Let $\tilde{F} \in (C_0(\mathbb{R}^n))^*$.

We now prove that F satisfies the hypothesis of RRT3 (local version).

Define:

$$F := \tilde{F}\big|_{C_c(\mathbb{R}^n)}.$$

$F : C_c(\mathbb{R}^n) \to \mathbb{R}$. is a linear functional.

Fix $K \subset \mathbb{R}^n$ compact. Then

$$\sup \{ F(f) \mid f \in C_c(\mathbb{R}^n),\ |f| \le 1,\ \operatorname{spt}(f) \subset K \}$$

$$\le \sup \{ \tilde{F}(f) \mid f \in C_0(\mathbb{R}^n),\ |f| \le 1,\ \operatorname{spt}(f) \subset K \}$$

$$= \|\tilde{F}\| < \infty$$

Hence:

$$\sup \{ F(f) \mid f \in C_c(\mathbb{R}^n),\ |f| \le 1,\ \operatorname{spt}(f) \subset K \} < \infty$$

for each compact set $K \subset \mathbb{R}^n$.

From RRT3 (local version):

$\exists\, \tilde{\mu}$, Radon outer measure on $\mathbb{R}^n$, non-negative,

$\exists\, \sigma : \mathbb{R}^n \to \mathbb{R}$ such that

$$|\sigma(x)| = 1 \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^n$$

and

$$\text{(A)} \quad \boxed{F(f) = \int_{\mathbb{R}^n} f\, \sigma\, d\tilde{\mu}, \quad \forall f \in C_c(\mathbb{R}^n)}$$

Define

$$\nu := \sigma \tilde{\mu}$$

Note: From the proof of RRT(3), local version, that:

$$\tilde{\mu}(\mathbb{R}^n) = \sup\{F(f) : f \in C_c(\mathbb{R}^n), |f| \le 1, \operatorname{Spt}(f) \subset \mathbb{R}^n\}$$

and hence

$$\tilde{\mu}(\mathbb{R}^n) < \infty$$

since

$$\sup\{F(f) : f \in C_c(\mathbb{R}^n), |f| \le 1, \operatorname{Spt}(f) \subset \mathbb{R}^n\}$$
$$\le \sup\{\tilde{F}(f) : f \in C_0(\mathbb{R}^n), |f| \le 1\} = \|\tilde{F}\| < \infty.$$

Claim: $\Psi(\mu) = \tilde{F}$

Let $f \in C_0(\mathbb{R}^n)$. Let $\{f_k\} \in C_c(\mathbb{R}^n)$ such that:

$$f_k \to f \quad \text{uniformly.}$$

Then, since $\tilde{F}$ is continuous:

$$\tilde{F}(f) = \lim_{k\to\infty} \tilde{F}(f_k)$$
$$= \lim_{k\to\infty} F(f_k); \quad F(f_k) = \tilde{F}(f_k)$$
$$= \lim_{k\to\infty} \int_{\mathbb{R}^n} f_k \sigma \, d\tilde{\mu}, \quad \text{by (A)}$$

Since $\int_{\mathbb{R}^n} |\sigma| \, d\tilde{\mu} < \infty$,

the Lebesgue - Dominated Convergence Theorem gives:

$$\tilde{F}(f) = \lim_{k \to \infty} \int_{\mathbb{R}^n} f_k \sigma \, d\tilde{\mu}$$

$$= \int_{\mathbb{R}^n} f \sigma \, d\tilde{\mu}$$

$$= \int_{\mathbb{R}^n} f \, d\nu$$

$$= \Psi(\nu)(f)$$

$$\therefore \quad \tilde{F}(f) = \Psi(\nu)(f), \quad \forall f \in C_0(\mathbb{R}^n)$$

$$\therefore \quad \tilde{F} = \Psi(\nu)$$

$\therefore$ Ψ is on-to.

Step 3 : Υ is an isometry;
i.e:

$$\|\Upsilon(\mu)\| = \|\mu\|$$

Fix $\mu \in \mathcal{M}(\mathbb{R}^n)$. Since $\Upsilon(\mu) \in (C_0(\mathbb{R}^n))^*$,

then Step 2 gives:

$\nu = \sigma\tilde{\mu}$, $\tilde{\mu}$ positive
$|\sigma(x)| = 1$ $\tilde{\mu}$-a.e. x.

such that

$$\Upsilon(\nu) = \Upsilon(\mu)$$

$$\therefore \int_{\mathbb{R}^n} f\,d\nu = \int_{\mathbb{R}^n} f\,d\mu, \quad \forall f \in C_0(\mathbb{R}^n)$$

$$\therefore \boxed{\int_{\mathbb{R}^n} f\,d(\nu-\mu) = 0, \quad \forall f \in C_0(\mathbb{R}^n)}$$

Let $E \subset \mathbb{R}^n$, bounded $|\nu-\mu|$-measurable. Let $\{f_k\}$ be a sequence of continuous functions with compact support such that

$$f_k(x) \to \chi_E(x) \quad |\nu-\mu|\text{-a.e. } x.$$

(This sequence can be constructed by applying Lusin's Thm to χ_E).

$\therefore \int_{\mathbb{R}^n} f_k \, d(\nu-\mu) = 0$

By Lebesgue Domitated Convergence Theorem:

$\int_{\mathbb{R}^n} \chi_E \, d(\nu-\mu) = 0$

$\therefore$

$\nu(E) = \mu(E)$ for every measurable set E. bounded

$\therefore \boxed{\nu = \mu}$

$$\begin{aligned}
\therefore \|\mu\| &= \|\nu\| \\
&= |\sigma\tilde{\mu}|(\mathbb{R}^n) \\
&= \int_{\mathbb{R}^n} |\sigma(x)| \, d\tilde{\mu} \\
&= \int_{\mathbb{R}^n} d\tilde{\mu} = \tilde{\mu}(\mathbb{R}^n) \\
&= \sup\{\tau(\mu)(f) : f \in C_c(\mathbb{R}^n), |f| \leq 1, \operatorname{spt}(f) \subset \mathbb{R}^n\} \\
&\leq \sup\{\tau(\mu)(f) : f \in C_0(\mathbb{R}^n), |f| \leq 1\} \\
&= \|\tau(\mu)\|
\end{aligned}$$

$\therefore \|\mu\| = \|\tau(\mu)\|, \quad \forall \mu \in \mathcal{M}(\mathbb{R}^n)$

Step 4: ψ is injective

Since ψ is linear, we only need to check:

$$\psi(\mu) = 0 \implies \mu \equiv 0$$

Suppose $\int_{\mathbb{R}^n} f \, d\mu = 0$, $\forall f \in C_0(\mathbb{R}^n)$

Let $\mu = \mu^+ - \mu^-$

$$\therefore \int_{\mathbb{R}^n} f \, d\mu^+ = \int_{\mathbb{R}^n} f \, d\mu^-, \quad \forall f \in C_0(\mathbb{R}^n)$$

Let E be a μ^+, μ^- measurable set. Then there exists a sequence (using Lusin's thm) $\{f_k\} \in C_c(\mathbb{R}^n)$ with

$$f_k \to \chi_E, \quad |\mu|\text{- almost everywhere}$$

$$\therefore \int_{\mathbb{R}^n} f_k \, d\mu^+ = \int_{\mathbb{R}^n} f_k \, d\mu^-$$

Letting $k \to \infty$:

$$\int_{\mathbb{R}^n} \chi_E \, d\mu^+ = \int_{\mathbb{R}^n} \chi_E \, d\mu^-$$

$$\therefore \mu^+(E) = \mu^-(E), \quad \forall E \text{ measurable.}$$

$$\therefore \mu^+ = \mu^-$$

$$\therefore \mu \equiv 0.$$

Thm: Let T be a distribution. Then

T is a Radon measure $\iff$ T is of order 0.

Proof:

We have already seen that if T is a Radon measure then T is a distribution of order 0.

Assume now that T is a distribution of order 0; i.e; $\forall K$ compact.

$$|T(\varphi)| \leq C(K)\, \|\varphi\|_{K;0}, \quad \varphi \in C_c^\infty, \ spt(\varphi) \subset K$$

Proceeding as in the Corollary of RRT3 given in an earlier version, we can extend $T: C_c^\infty(\Omega) \to \mathbb{R}$ to a linear functional:

$$T^*: C_c(\Omega) \to \mathbb{R}$$

$$|T^*(\varphi)| \leq C(K)\, \|\varphi\|_{K;0}, \quad \varphi \in C_c(\Omega), \ spt(\varphi) \subset K.$$

If $|\varphi| \leq 1$; this says:

$$\sup\{ T^*(\varphi) : |\varphi| \leq 1, \ spt(\varphi) \subset K, \ \varphi \in C_c(\Omega)\} < \infty,$$

for every compact set K. Then RRT3, local version, gives a (signed) Radon measure μ such that:

$T^*(\varphi) = \int \varphi \, d\mu$, $\forall \varphi \in C_c(\Omega)$. In particular, $T(\varphi) = \int \varphi \, d\mu$, $\varphi \in C_c^\infty(\Omega)$.

<u>Definition</u> : A distribution on Ω is positive if $T(\varphi) \geq 0$ for all test functions $\varphi \in \mathcal{D}(\Omega)$ satisfying $\varphi \geq 0$.

<u>Thm</u> : A distribution T on Ω is positive if and only if T is a positive measure.

<u>Proof</u>:

If T is a positive measure μ then clearly the associated distribution of order 0:

$$T(f) = \int_{\mathbb{R}^n} f \, d\mu \qquad \forall f \in \mathcal{D}(\Omega)$$

satisfies:

$$T(f) \geq 0 \quad \text{if} \quad f \geq 0.$$

The other direction is a Corollary of RRT3, which was proved in an earlier lesson.

Remark on the RRT3:

RRT3, Global version, can be restated by saying that the dual of the Banach space $C_0(\mathbb{R}^n)$ is the space $\mathcal{M}(\mathbb{R}^n)$ of <u>finite</u> Radon measures (i.e. $|\mu|(\mathbb{R}^n) < \infty$) under the pairing

$$(f, \mu) = \int_{\mathbb{R}^n} f \, d\mu .$$

$$\mathcal{M}(\mathbb{R}^n) \sim C_0(\mathbb{R}^n)^*$$

RRT3, Local version, can be restated by saying that the dual of the <u>locally convex</u> space $C_c(\mathbb{R}^n)$ is the space $\mathcal{M}_{loc}(\mathbb{R}^n)$ of Radon measures ($|\mu|(K) < \infty$, $\forall K \subset \mathbb{R}^n$ compact), under the pairing

$$(f, \mu) = \int_{\mathbb{R}} f \, d\mu .$$

$$\mathcal{M}_{loc}(\mathbb{R}^n) \sim (C_c(\mathbb{R}^n))^* .$$

A diagram of important concepts.

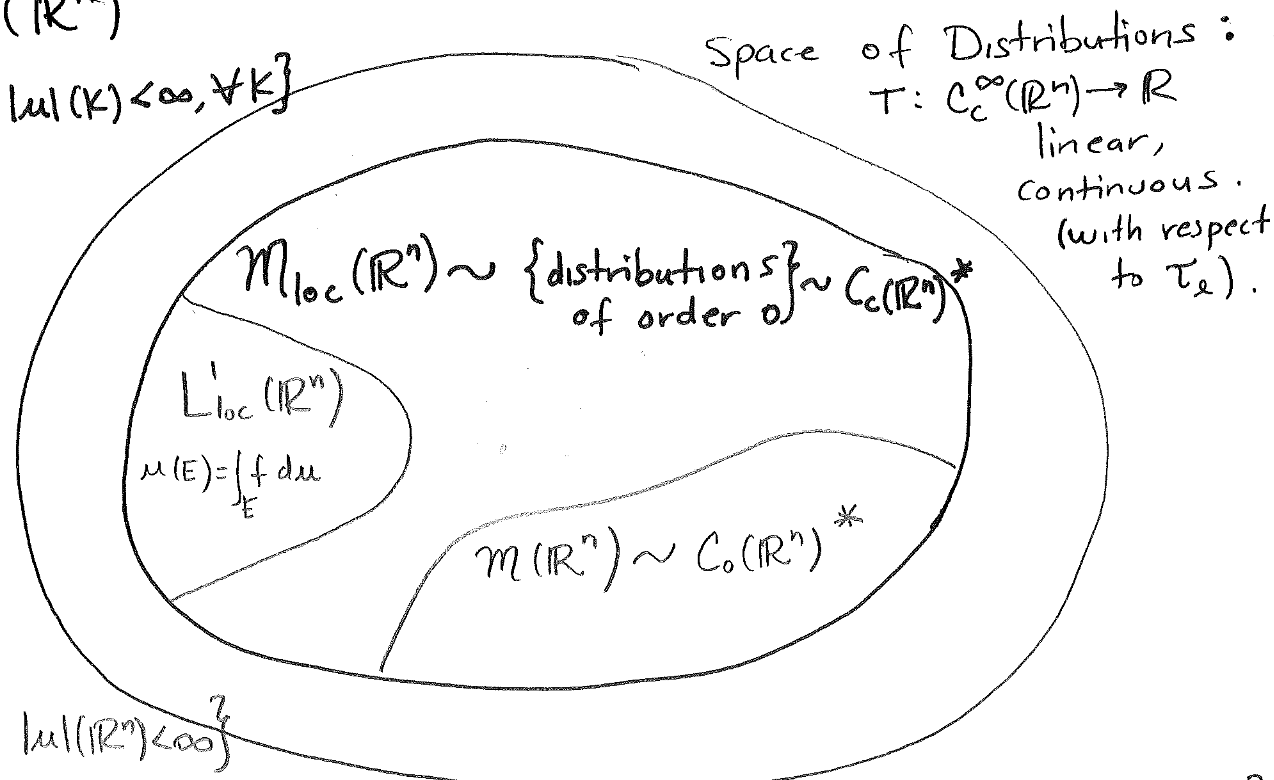
$C_c^\infty(\mathbb{R}^n) \subset C_c(\mathbb{R}^n)$

These space are equipped with the "locally convex" topology τ_ℓ.

- A distribution $T: C_c^\infty(\mathbb{R}^n) \to \mathbb{R}$ is a linear functional that is continuous (with respect to τ_ℓ).
- $\{T : T \text{ is a distribution of order } 0\}$ can be identified with $\mathcal{M}_{loc}(\mathbb{R}^n)$
- $\mathcal{M}_{loc}(\mathbb{R}^n)$ can be identified with $C_c(\mathbb{R}^n)^*$

$\therefore$ $\{T : T \text{ is a dist. of order } 0\}$ can be identified (through an extension) with $C_c(\mathbb{R}^n)^*$

$\mathcal{M}_{loc}(\mathbb{R}^n) = \{\mu : |\mu|(K) < \infty, \forall K\}$



$\mathcal{M}(\mathbb{R}^n) = \{\mu : |\mu|(\mathbb{R}^n) < \infty\}$

$\mathcal{M}(\mathbb{R}^n) \subset \mathcal{M}_{loc}(\mathbb{R}^n)$

$L^1_{loc}(\mathbb{R}^n) \subset \mathcal{M}_{loc}(\mathbb{R}^n) \subset \{\text{Distributions}\}$