- **Note:** If

  $f = g$ λ-almost everywhere

  Then

  $$T_f' = T_g',$$

  where T_f, T_g are the distributions corresponding to f and g respectively.

Def: The essential variation of a function f defined on (a,b) is:

$$\operatorname{ess} V_a^b f = \sup \left\{ \sum_{i=1}^{k} |f(t_{i+1}) - f(t_i)| \right\}$$

where the supremum is taken over all finite partitions $a < t_1 < \dots < t_{k+1} < b$ such that each t_i is a point of approximate continuity of f.

Note: If

$f = g$ λ-almost everywhere

Then

$$\operatorname{ess} V_a^b f = \operatorname{ess} V_a^b g$$

Def: Let μ be a signed finite Radon measure in the open set Ω. The norm of μ, is defined as:

$$\|\mu\| := |\mu|(\Omega)$$

We have:

Thm 1: Suppose $f \in L^1(a,b)$. Then;

$f' = \mu$ (in the sense of distributions), $|\mu|(a,b) < \infty$ $\Longleftrightarrow$ $\text{ess } V_a^b f < \infty$

Moreover:

$$\|f'\| = |\mu|(a,b) = \text{ess } V_a^b F$$

The proof of this Theorem uses the fact that, if μ is a signed finite measure in Ω, then:

$$\boxed{|\mu|(\Omega) = \sup\left\{ \int_\Omega \varphi \, d\mu : \varphi \in C_c(\Omega), |\varphi| \leq 1 \right\}} \quad (1)$$

(1) is clear from the proof of RRT3, Local & Global version. However, we present again the proof next:

<u>Lemma 1</u>: Let μ be a (signed) finite Radon measure in a set $E \subset \mathbb{R}^n$. Then, for every open set $\Omega \subset E$:

$$|\mu|(\Omega) = \sup \left\{ \int_\Omega f \, d\mu : f \in C_c(\Omega),\ |f| \leq 1 \right\}.$$

<u>Note:</u> If $E \subsetneq \mathbb{R}^n$, we consider E as a metric space endowed with the induced topology from $\mathbb{R}^n$. Thus, $\Omega \subset E$ is open in the relative topology

<u>Proof:</u>

Since $\mu \ll |\mu|$, Radon-Nikodym Theorem yields:

$\exists\ f \in L^1(E)$ s.t. $\mu = f|\mu|$. That is:

$$\mu(A) = \int_A f \, d|\mu|, \quad A \subset E \text{ measurable.}$$

Then:

$$|\mu|(A) = \int_A |f| \, d|\mu|, \quad A \subset E$$

and this implies $|f(x)| = 1$, $|\mu|$-a.e. x.

We have therefore:

$$\boxed{\mu = f|\mu|, \quad |f| = 1 \text{ a.e. } x} \qquad (2)$$

Choose now a sequence $\{f_K\} \subset C_c(\Omega)$ such that:

$$f_K(x) \to f(x), \quad |\mu|\text{-a.e. } x, \quad |f_K| \leq 1.$$

(We can choose f_K by applying Lusin's Theorem to f).

Since $|\mu|(\Omega) < \infty$, we can apply the Lebesgue Dominated convergence Theorem:

$$\lim_{K\to\infty} \int_\Omega f_K \, d\mu = \lim_{K\to\infty} \int_\Omega f_K \cdot f \, d|\mu|; \quad \text{by (2)}$$

$$= \int_\Omega f \cdot f \, d|\mu|$$

$$= \int_\Omega f^2 \, d|\mu|$$

$$= \int_\Omega d|\mu|; \quad \text{by (2)}$$

$$= |\mu|(E)$$

Therefore; for every $\varepsilon > 0$, $\exists N(\varepsilon)$ s.t.

$$|\mu|(E) \leq \int_\Omega f_K \, d\mu + \varepsilon, \quad \forall K \geq N(\varepsilon)$$

$$\leq \sup\left\{ \int_\Omega f \, d\mu, \ f \in C_c(\Omega), \ |f| \leq 1 \right\} + \varepsilon$$

$$\therefore \boxed{|\mu|(E) \leq \sup\left\{ \int_\Omega f \, d\mu, \ f \in C_c(\Omega), \ |f| \leq 1 \right\}} \quad (3)$$

(since ε is arbitrary)

For the reverse inequality:

$$\int_\Omega f\,d\mu \le \int_\Omega |f|\,d|\mu|$$

$$\le \|f\|_\infty |\mu|(\Omega)$$

$$\therefore \int_\Omega f\,d\mu \le |\mu|(\Omega), \quad f\in C_c(\Omega),\ |f|\le 1$$

Hence:

$$\boxed{\sup\left\{\int_\Omega f\,d\mu : f\in C_c(\Omega),\ |f|\le 1\right\} \le |\mu|(\Omega)} \quad (4).$$

Actually, notice that:

$|\mu|(\Omega) = \sup\left\{\int_\Omega f\,d\mu : f\in C_c^\infty(\Omega),\ |f|\le 1\right\}$;

Since $C_c^\infty(\Omega)$ is dense in $C_c(\Omega)$.

Also:

$$|\mu|(\Omega) = \sup\left\{\int_\Omega f\,d\mu : f\in C_0(\Omega),\ |f|\le 1\right\}$$

$$= \sup\left\{\int_\Omega f\,d\mu : f\in C(\Omega),\ |f|\le 1\right\}$$

In fact, for $C_0(\Omega)$ or $C(\Omega)$ we can prove (3) and (4) following the same proof as in Lemma 1;

since: $|\mu|(E) \le \int_\Omega f_k\,d\mu + \varepsilon,\ \forall k\ge N(\varepsilon)$

$$\le \sup\left\{\int_\Omega f\,d\mu,\ f\in C_0(\Omega),\ |f|\le 1\right\} + \varepsilon$$

$$\le \sup\left\{\int_\Omega f\,d\mu,\ f\in C(\Omega),\ |f|\le 1\right\} + \varepsilon$$

At this point, we would like to introduce another version of RRT3:

RRT3 (Compact version). Let $K \subset \mathbb{R}^n$ a compact set. We know that:

$$C(K) = \{f : K \to \mathbb{R},\ f \text{ is continuous}\}$$

is a Banach space with the sup norm.

Then, the map:

$$\Upsilon : \mathcal{M}(K) \longrightarrow (C(K))^*$$

$$\Upsilon(\mu) = \int_K f \, d\mu, \quad f \in C(K)$$

is an isometric isomorphism.

Recall:

$$\mathcal{M}(K) = \{\mu : |\mu|(K) < \infty,\ \mu \text{ Borel regular}\}$$

Note that, since K is compact:

$$C_c(K) = C_0(K) = C(K).$$