Change of Variable Formula

We want to show the following:

<u>Change of Vanable Formula</u>: Let $T: \mathbb{R}^n \to \mathbb{R}^n$ a Lipschitz map, and $f \in L^1(\mathbb{R}^n)$. If $E \subset \mathbb{R}^n$ is Lebesgue measurable and T is injective on E, then:

$$\int_{T(E)} f(y)\, d\lambda(y) = \int_E f \circ T(x)\, |JT(x)|\, d\lambda(x)$$

<u>Note:</u> Since T is Lipschitz, then it is differentiable λ-a.e. Moreover, for λ-a.e. x we have:

$$dT(x) = \begin{bmatrix} \frac{\partial T^1}{\partial x_1} & \cdots & \frac{\partial T^1}{\partial x_n} \\ \vdots & & \\ \frac{\partial T^n}{\partial x_1} & \cdots & \frac{\partial T^n}{\partial x_n} \end{bmatrix}$$

where $T = (T^1, T^2, \ldots, T^n)$.

We then write, for λ-a.e. x:

$JT(x) :=$ determinant of $dT(x)$.

The Change of Variables formula is a consequence of the Area formula:

Thm (Area formula): Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be Lipschitz. Then for each Lebesgue measurable set $E \subset \mathbb{R}^n$,

$$\int_E |JT(x)| \, d\lambda(x) = \int_{\mathbb{R}^n} N(T, E, y) \, d\lambda(y),$$

where $N(T, E, y)$ is the (possibly infinite) number of points in $E \cap T^{-1}(y)$

Claim: Area formula $\Rightarrow$ Change of variable formula

Suppose first that $f = \chi_A$, A Lebesgue measurable. Then:

$$\chi_A \circ T = \chi_{T^{-1}(A)}$$

$$[\, x \in T^{-1}(A) \iff T(x) \in A \,]$$

Hence:

$$\int_{\mathbb{R}^n} f\circ T(x)\, |JT(x)|\, d\lambda(x) =$$

$$= \int_{\mathbb{R}^n} \chi_{T^{-1}(A)}(x)\, |JT(x)|\, d\lambda(x)$$

$$= \int_{T^{-1}(A)} |JT(x)|\, d\lambda(x)$$

$$= \int_{\mathbb{R}^n} N(T, T^{-1}(A), y)\, d\lambda(y) \quad ; \text{ by Area formula}$$

$$= \int_{\mathbb{R}^n} \chi_A(y)\, N(T, \mathbb{R}^n, y)\, d\lambda(y)$$

$$= \int_{\mathbb{R}^n} f(y)\, N(T, \mathbb{R}^n, y)\, d\lambda(y)$$

Clearly, this holds whenever f is a simple function. If $f \in L^1(\mathbb{R}^n)$, Then $f = f^+ - f^-$ and f^+, f^- can be approximated by sequences of simple functions:

$$g_i \uparrow f^+ \quad , \quad h_i \uparrow f^-$$

Hence:

$$\int_{\mathbb{R}^n} g_i \circ T(x)\, |JT(x)|\, d\lambda(x) = \int_{\mathbb{R}^n} g_i(y)\, N(T, \mathbb{R}^n, y)\, d\lambda(y)$$

and

$$\int_{\mathbb{R}^n} h_i \circ T(x)\, |JT(x)|\, d\lambda(x) = \int_{\mathbb{R}^n} h_i(y)\, N(T, \mathbb{R}^n, y)\, d\lambda(y)$$

Letting $i \to \infty$ and applying the Monotone Convergence Theorem we get:

$$\int_{\mathbb{R}^n} f^+ \circ T(x)\, |JT(x)|\, d\lambda(x) = \int_{\mathbb{R}^n} f^+(y)\, N(T, \mathbb{R}^n, y)\, d\lambda(y)$$

and

$$\int_{\mathbb{R}^n} f^- \circ T(x)\, |JT(x)|\, d\lambda(x) = \int_{\mathbb{R}^n} f^-(y)\, N(T, \mathbb{R}^n, y)\, d\lambda(y).$$

We conclude:

$$\int_{\mathbb{R}^n} (f^+ - f^-) \circ T(x)\, |JT(x)|\, d\lambda(x) = \int_{\mathbb{R}^n} (f^+ - f^-)(y)\, N(T, \mathbb{R}^n, y)\, d\lambda(y)$$

or

$$\int_{\mathbb{R}^n} f \circ T(x)\, |JT(x)|\, d\lambda(x) = \int_{\mathbb{R}^n} f(y)\, N(T, \mathbb{R}^n, y)\, d\lambda(y),$$

for every $f \in L^1(\mathbb{R}^n)$. In particular, with $\chi_{T(E)} f$ instead of f (and hence $N(T, \mathbb{R}^n, y) \equiv 1$ on $T(E)$), we obtain the Change of Variables formula:

$$\boxed{\int_E f \circ T(x)\, |JT(x)|\, d\lambda(x) = \int_{T(E)} f(y)\, d\lambda(y).} \quad (A)$$

In the Change of Variable formula, if we put $f = \chi_{T(E)}$, E Lebesgue measurable

then the formula reduces to:

$$(1)\quad \int_{\mathbb{R}^n} \chi_{T(E)} \circ T(x)\, |JT(x)|\, d\lambda(x) = \int_{\mathbb{R}^n} \chi_{T(E)}(y)\, d\lambda(y)$$

Note:

$$x \notin E \iff T(x) \notin T(E) \iff \chi_{T(E)} \circ T(x) = 0$$

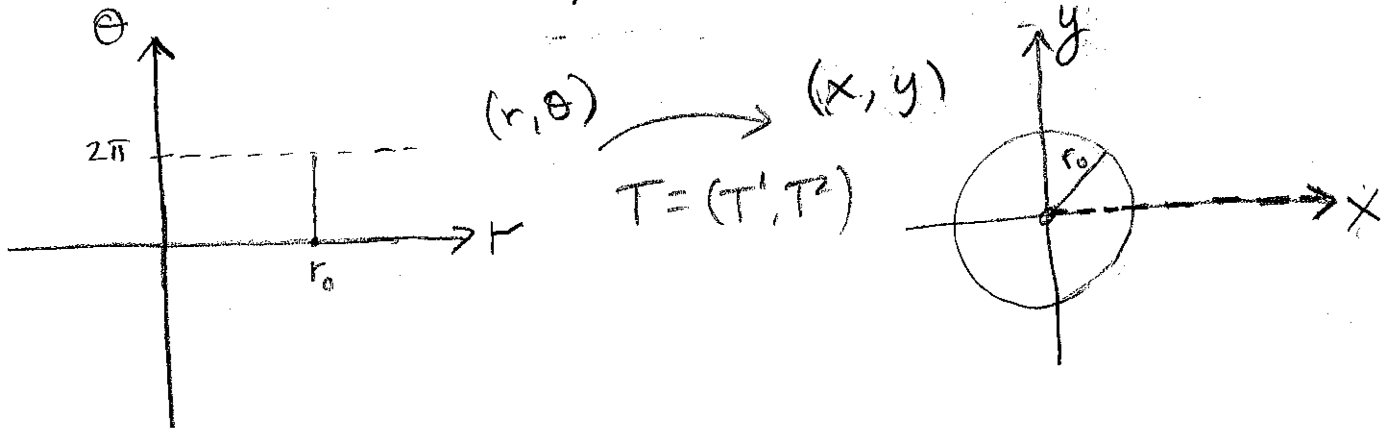
$$x \in E \iff T(x) \in T(E) \iff \chi_{T(E)} \circ T(x) = 1$$

Thus, (1) reduces to:

$$\boxed{\int_E |JT(x)|\, d\lambda(x) = \lambda(T(E))}$$

Ex: Spherical Coordinates in $\mathbb{R}^n$.

We consider first the case $n=2$:



$$T^1(r,\theta) = x = r\cos\theta$$
$$T^2(r,\theta) = y = r\sin\theta$$

We have:
$$T : (0,\infty)\times(0,2\pi) \longrightarrow \mathbb{R}^2 \setminus N,$$
where $N = \{(x,0) : x \geq 0\}$

T is 1-1, on-to, and C^∞.

The Change of variable formula gives, for any $A \subset (0,\infty)\times(0,2\pi)$:

$$\int_{T(A)} f(x,y)\, d\lambda(x,y) = \int_A f\circ T(r,\theta)\ |JT(r,\theta)|\ d\lambda(r,\theta)$$
$$= \int_A f\circ T(r,\theta)\, r\, dr\, d\theta$$

11.231

If $A = (0,1) \times (0, 2\pi)$,

then $T(A) = \{(x,y) : x^2 + y^2 < 1\} \setminus N$

$$= B(0,1) \setminus N$$

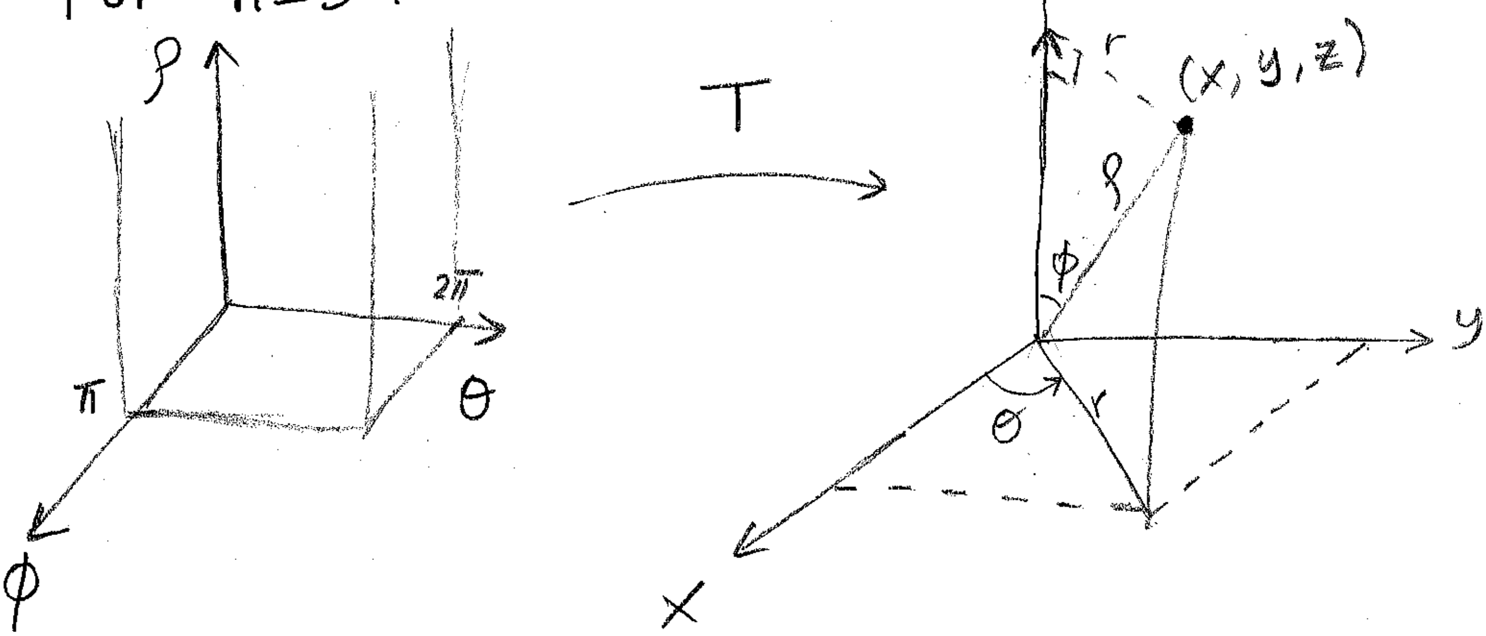
Hence:

$$\int_{B(0,1)} f(x,y)\, dx\, dy = \int_0^{2\pi} \int_0^1 f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

<u>Note</u>: $\lambda(N) = 0$ and hence:

$$\int_{B(0,1)} f\, d\lambda(x,y) = \int_{T(A)} f\, d\lambda(x,y).$$

For $n=3$:



$$(\rho, \phi, \theta) \xrightarrow{T} (x, y, z)$$

$$T = (T^1, T^2, T^3)$$

$$T^1(\rho, \phi, \theta) = x = r\cos\theta = \rho \sin\phi \cos\theta$$

$$T^2(\rho, \phi, \theta) = y = r\sin\theta = \rho\sin\phi\sin\theta$$

$$T^3(\rho, \phi, \theta) = z = \rho\cos\phi$$

We have:

$T: (0,\infty) \times (0,\pi) \times (0, 2\pi) \to \mathbb{R}^3 \setminus N$

is 1-1 and on-to (Note: $\lambda(N) = 0$).

The Change of vanable formula gives, for any $A \subset (0,\infty) \times (0,\pi) \times (0,2\pi)$:

$$\int_{T(A)} f(x,y,z)\, d\lambda(x,y,z) = \int_A f\circ T(\rho,\phi,\theta)\, |JT(\rho,\phi,\theta)|\, d\lambda(\rho,\phi,\theta)$$

$$= \int_A f\circ T(\rho,\phi,\theta)\, \rho^2 \sin\phi \, d\lambda(\rho,\phi,\theta)$$

In particular, if $A = (0,1) \times (0,\pi) \times (0,2\pi)$

we obtain:

$T(A) = \{(x,y,z) : x^2+y^2+z^2 < 1\} \setminus N$

Hence:

$$\int_{B(0,1)} f(x,y,z)\,dx\,dy\,dz = \int_0^1 \int_0^{\pi} \int_0^{2\pi} f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\, \rho^2 \sin\phi \, d\theta\, d\phi\, d\rho$$

Prove the following:

1.- Use the Change of Variable formula to show that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

2.- Prove that $|JT(\rho, \phi, \theta)| = \rho^2 \sin\phi$

3.- Use the Change of variable formula to compute the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.