Sketch of Proof of: Area Formula.

Area Formula:
Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be Lipschitz.
Then, for each Lebesgue measurable set $E \subset \mathbb{R}^n$:

$$\int_E |JT(x)| \, d\lambda(x) = \int_{\mathbb{R}^n} N(T, E, y) \, d\lambda(y).$$

__Proof__:

1.- In view of Rademacher's Theorem, we can assume:

$dT(x)$ exists $\forall x \in E$

> Lemma 1: $T: \mathbb{R}^n \to \mathbb{R}^n$ Lipschitz.
> $A = \{x \in \mathbb{R}^n ; \ dT(x) \text{ exists and } JT(x) = 0\}$
> $\Rightarrow \lambda(T(A)) = 0$

In view of Lemma 1 we can assume:

$$|JT(x)| \neq 0, \quad \forall x \in E$$

Since we can write $\mathbb{R}^n = \bigcup E_i$, $E_i \cap E_j \neq 0$, $\lambda(E_i) < \infty$, we can assume also that $\lambda(E) < \infty$.

(2).- Let:

$$\mathcal{C}_j = \{ Q \mid Q = [a_1, b_1) \times \cdots \times [a_n, b_n), \quad a_i = \frac{c_i}{j}, \quad b_i = \frac{c_i + 1}{j}, \quad c_i \text{ integers}, \quad i = 1, 2, \ldots, n \}$$

and note:

$$\mathbb{R}^n = \bigcup_{Q_i^j \in \mathcal{C}_j} Q_i^j$$

We have

Thm 1: Given $t > 1$, $\exists \{B_k\}_{k=1}^{\infty}$ such that:

$$E = \bigcup_{k=1}^{\infty} B_k, \quad B_k \text{ Borel}$$

with:

$$T_k := T|_{B_k} \text{ is } 1\text{-}1.$$

For each k, $\exists L_k : \mathbb{R}^n \to \mathbb{R}^n$ linear, non-singular such that:

$$t^{-n} |\det L_k| \le |JT(x)| \le t^n |\det L_k|, \quad x \in B_k$$

$$\underline{\mathrm{Lip}(T_k \circ L_k^{-1}) \le t}, \quad \underline{\mathrm{Lip}(L_k \circ T_k^{-1}) \le t}$$

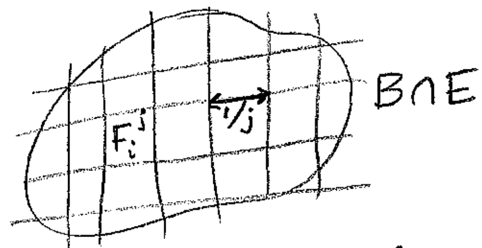
Select some set B_K:

Let

$$B := B_K.$$

Define:

$$F_i^j = B \cap E \cap Q_i^j, \quad Q_i^j \in \mathcal{C}^j$$



$$B \cap E = \bigcup_{i=1}^{\infty} F_i^j$$

T Lipschitz $\Rightarrow T(F_i^j)$ λ-measurable

Define:

$$g_j(y) = \sum_{i=1}^{\infty} \chi_{T(F_i^j)}(y)$$

Note:

$g_j(y) =$ Number of sets $\{F_i^j\}$ such that $F_i^j \cap T^{-1}(y) \neq \emptyset$.

and:

$$\boxed{\lim_{j \to \infty} g_j(y) = N(T, B \cap E, y).}$$

$$\left(y \in T(F_i^j) \iff y = T(z),\ z \in F_i^j \right.$$
$$\left. \iff T^{-1}(y) \cap F_i^j \neq \emptyset \right)$$

Monotone Convergence Theorem $\Rightarrow$

$$\lim_{j\to\infty} \int_{\mathbb{R}^n} g_j(y) = \int_{\mathbb{R}^n} N(T, B\cap E, y)$$

$$\therefore \lim_{j\to\infty} \int_{\mathbb{R}^n} \sum_{i=1}^{\infty} \chi_{T(F_i^j)}(y)\, d\lambda(y) = \int_{\mathbb{R}^n} N(T, B\cap E, y)\, d\lambda$$

$$\therefore \lim_{j\to\infty} \sum_{i=1}^{\infty} \lambda(T(F_i^j)) = \int_{\mathbb{R}^n} N(T, B\cap E, y)\, d\lambda(y)$$

(3) Let L_k and T_k as in Theorem 1.
Recall that $B = B_k$. The following is true:

Lemma 2: $L : \mathbb{R}^n \to \mathbb{R}^n$, linear, non-singular
$\Rightarrow \lambda(L(E)) = |\det L|\, \lambda(E)$

We have:

$$\lambda(T(F_i^j)) = \lambda(T_k(F_i^j)) = \lambda[(T_k \circ L_k^{-1} \circ L_k)(F_i^j)] \le t^n \lambda(L_k(F_i^j))$$

$$\lambda(L_k(F_i^j)) = \lambda[(L_k \circ T_k^{-1} \circ T_k)(F_i^j)] \le t^n \lambda[T_k(F_i^j)] = t^n \lambda[T(F_i^j)]$$

Here we have used Thm 1 and the fact that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz then

$$\lambda(f(E)) \le (\mathrm{Lip}(f))^n \lambda(E)$$

Using Lemma 2 and Thm 1:

$$t^{-2n}\lambda(T(F_i^j)) \leq t^{-n}\lambda[L_K(F_i^j)] = t^{-n}\,|\det L_K|\,\lambda(F_i^j)$$

$$\leq \int_{F_i^j} |JT(x)|\,d\lambda(x)$$

$$\leq t^{n}|\det L_K|\,\lambda(F_i^j) = t^{n}\lambda[L_K(F_i^j)] \leq t^{2n}\lambda[T(F_i^j)]$$

$\therefore$

$$t^{-2n}\lambda[T(F_i^j)] \leq \int_{F_i^j} |JT(x)| \leq t^{2n}\lambda[T(F_i^j)], \text{ fixed } j$$

$\Rightarrow$

$$t^{-2n}\sum_{i=1}^{\infty}\lambda[T(F_i^j)] \leq \int_{B\cap E} |JT(x)|\,d\lambda(x) \leq t^{2n}\sum_{i=1}^{\infty}\lambda[T(F_i^j)]$$

By ② and letting $j\to\infty$:

$$t^{-2n}\int_{\mathbb{R}^n} N(T, B\cap E, y)\,d\lambda(y) \leq \int_{B\cap E} |JT(x)|\,d\lambda(x) \leq t^{2n}\int_{\mathbb{R}^n} N(T, B\cap E, y)\,d\lambda(y)$$

Since t is any number larger than 1; letting $t\to 1$:

$$\int_{\mathbb{R}^n} N(T, B\cap E, y)\,d\lambda(y) \leq \int_{B\cap E} |JT(x)|\,d\lambda(x) \leq \int_{\mathbb{R}^n} N(T, B\cap E, y)\,d\lambda(y)$$

$$\therefore \int_{B\cap E} |JT(x)|\,d\lambda(x) = \int_{\mathbb{R}^n} N(T, B\cap E, y)\,d\lambda(y)$$

B was defined as an arbitrary set of the sequence $\{B_K\}$ given by Theorem 1. Thus, for every K:

$$\int_{B_K \cap E} |JT(x)| \, d\lambda(x) = \int_{\mathbb{R}^n} N(T, B_K \cap E, y) \, d\lambda(y)$$

Since:

$$E = \bigcup_{K=1}^{\infty} B_K$$

and the $\{B_K\}$ are disjoint, by additivity we obtain

$$\boxed{\int_E |JT(x)| \, d\lambda(x) = \int_{\mathbb{R}^n} N(T, E, y) \, d\lambda(y)}.$$