Derivatives of Distributions

Recall that if $f \in L^1_{loc}(\Omega)$ then f can be regarded as a distribution, say T_f, given by:

$$T_f : C_c^\infty(\Omega) \to \mathbb{R}$$
$$T_f(\varphi) = \int_\Omega f\varphi \, d\lambda .$$

Let α be any multi-index. Then the α^{th} derivative of the distribution T is another distribution defined by:

$$D^\alpha T : C_c^\infty(\Omega) \to \mathbb{R}$$
$$D^\alpha T(\varphi) = (-1)^{|\alpha|} T(D^\alpha \varphi).$$

Thus, if $f, g_\alpha \in L^1_{loc}(\Omega)$ we say that

"$\underline{D^\alpha f = g_\alpha}$"

A) if: $\underline{\int_\Omega f \, D^\alpha \varphi \, d\lambda = (-1)^{|\alpha|} \int_\Omega \varphi \, g_\alpha \, d\lambda}$, $\varphi \in \mathcal{D}(\Omega)$

We say: "The α derivative of f, in the sense of distributions, is equal to g_α"

Sobolev spaces:

<u>Definition</u>: Let $\Omega \subset \mathbb{R}^n$ an open set. For $1 \leq p \leq \infty$ and k a nonnegative integer, we say that f belongs to the Sobolev space:

$$W^{k,p}(\Omega)$$

if $D^\alpha f \in L^p(\Omega)$ for each multi-index α with $|\alpha| \leq k$. In particular, this implies that:

$$W^{k,p}(\Omega) \subset L^p(\Omega).$$

Similarly, the space:

$$W^{k,p}_{loc}(\Omega)$$

Consists of all f with $D^\alpha f \in L^p_{loc}(\Omega)$ for $|\alpha| \leq k$.

Recall that if $f \in L^1_{loc}(\Omega)$, the distribution corresponding to f, denoted as T_f is defined as:

$$T_f : C_c^\infty(\Omega) \to \mathbb{R}$$

$$T_f(\varphi) = \int_\Omega f \varphi \, d\lambda$$

and the derivatives of a distribution T is defined as:

$$\frac{\partial T}{\partial x_i}(\varphi) = -T\left(\frac{\partial \varphi}{\partial x_i}\right), \quad i = 1, 2, \ldots n$$

In particular:

$$\frac{\partial}{\partial x_i} T_f(\varphi) = -T_f(\varphi)$$

$$= -\int_\Omega f \frac{\partial \varphi}{\partial x_i} \, d\lambda$$

Thus $\boxed{\frac{\partial}{\partial x_i} T_f = g_i}$ means

(B) $\boxed{-\int_\Omega f \frac{\partial \varphi}{\partial x_i} \, d\lambda = \int_\Omega g_i \varphi \, d\lambda, \quad \forall \varphi \in \mathcal{D}(\Omega)}$

(A) generalizes (B) for any multiindex α.

In order to motivate the definition of the distributional partial derivative, we recall the classical Gauss-Green Theorem:

<u>Thm</u> <u>Gaus-Green Theorem</u>: Suppose U is an open set with smooth boundary and let $\nu(x)$ denote the unit exterior normal to U at $x \in \partial U$. If $V: \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 vector field, then:

$$\int_U \operatorname{div} V \, d\lambda = \int_{\partial U} V(x) \cdot \nu(x) \, dH^{n-1}(x)$$

where $\operatorname{div} V$, the divergence of $V = (V^1, \dots, V^n)$, is defined by

$$\operatorname{div} V = \sum_{i=1}^{n} \frac{\partial V^i}{\partial x_i}$$

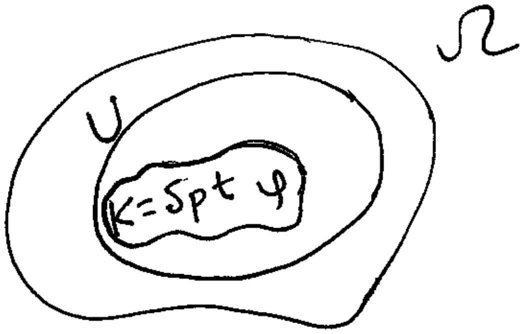
Ex: Let $f:\Omega \to \mathbb{R}$ C^1

$\varphi \in C_c^\infty(\Omega)$

$V: \mathbb{R}^n \to \mathbb{R}^n \quad V = (0,0,\dots,\underbrace{f\varphi}_{i\text{th coordinate}},\dots 0)$

Then:

$$\operatorname{div} V = \frac{\partial (f\varphi)}{\partial x_i} = f\frac{\partial \varphi}{\partial x_i} + \varphi \frac{\partial f}{\partial x_i}$$



$\exists\ U$, smooth boundary such that

$$K \subset U \subset \overline{U} \subset \Omega$$

We can then integrate by parts in U:

$$\int_\Omega \operatorname{div} V\, d\lambda = \int_U \operatorname{div} V\, d\lambda = \int_{\partial U} V\cdot \nu \, d\mathcal{H}^{n-1} = 0,$$

Since $V \equiv 0$ on ∂U

$$\therefore \int_U \operatorname{div} V\, d\lambda = \int_U f\frac{\partial \varphi}{\partial x_i} + \varphi\frac{\partial f}{\partial x_i}\, d\lambda = 0 \implies$$

(C) $\boxed{\int_\Omega f\frac{\partial \varphi}{\partial x_i}\, d\lambda = -\int_\Omega \frac{\partial f}{\partial x_i}\varphi}$, which is (B) in case $f \in C^1(\Omega)$

Ex: Let $f \in W^{k,p}(\Omega)$

Then (C) is also valid; that is:

$$\int_\Omega f \frac{\partial \varphi}{\partial x_i} \, d\lambda = - \int_\Omega \frac{\partial f}{\partial x_i} \varphi \, d\lambda,$$

for all $\varphi \in C_c^\infty(\Omega)$, by the definition of the distributional derivative.

The Sobolev norm of $f \in W^{1,p}(\Omega)$ is defined by:

$$(1) \quad \|f\|_{1,p;\Omega} := \|f\|_{p;\Omega} + \sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} \right\|_{p;\Omega}$$

for $1 \le p < \infty$ and

$$\|f\|_{1,\infty;\Omega} := \operatorname*{ess\,sup}_{\Omega} \left(|f| + \sum_{i=1}^{n} \left| \frac{\partial f}{\partial x_i} \right| \right)$$

Ex: $W^{1,p}(\Omega)$ is a Banach space with these norms, $1 \le p \le \infty$

Remark: When $1 \le p < \infty$, it can be shown (Exercise 11.6) that the norm:

$$\|f\| := \|f\|_{p;\Omega} + \left(\sum_{i=1}^{n} \|D_i f\|_{p;\Omega}^{p} \right)^{1/p}$$

is equivalent to:

$$\|f\| := \|f\|_{p;\Omega} + \sum_{i=1}^{n} \|D_i f\|_{p;\Omega}$$

Ex: $W^{1,p}(\Omega)$ is reflexive if $1 < p < \infty$.

Define:

$$P : W^{1,p}(\Omega) \to \Pi_{i=1}^{n+1} L^p(\Omega) \quad \text{as}$$

$$P(f) = \left(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$$

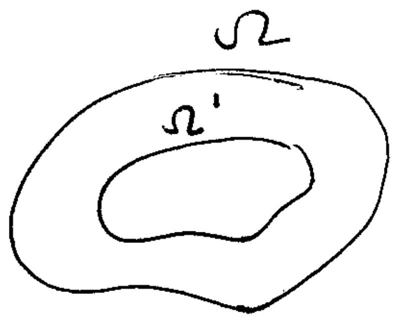
P is an isometric isomorphism of $W^{1,p}(\Omega)$ onto a subspace W of this Cartesian Product. (See Exercise 11.7). Since $W^{1,p}(\Omega)$ is a Banach space, W is a closed subspace. From Exercises 8.2 and 8.21 we have that $\Pi_{i=1}^{n+1} L^p(\Omega)$ is a Reflexive Banach space, and hence W is reflexive too. Therefore $W^{1,p}(\Omega)$ is reflexive.

Thm: Suppose $f \in W^{1,p}(\Omega)$, $1 \le p < \infty$. Then

$$\lim_{\varepsilon \to 0} \|f_\varepsilon - f\|_{1,p;\Omega'} = 0$$

whenever $\Omega' \subset\subset \Omega$; i.e: $\overline{\Omega'} \subset \Omega$, $\overline{\Omega'}$ compact.

Proof:



$\overline{\Omega'} \subset \Omega$

$\overline{\Omega'}$ compact

Let ε_0 s.t:
$\varepsilon_0 < \operatorname{dist}(\Omega', \Omega)$.

For $\varepsilon < \varepsilon_0$, we can differentiate under the integral sign (we justify this in the same way we showed earlier that $\frac{\partial}{\partial x_i}(\varphi_\varepsilon * f) = \frac{\partial \varphi_\varepsilon}{\partial x_i} * f$) to obtain for $x \in \Omega'$ and $1 \le i \le n$:

$$\begin{aligned}
\frac{\partial f_\varepsilon}{\partial x_i}(x) &= \frac{1}{\varepsilon^n} \int_\Omega \frac{\partial \varphi}{\partial x_i}\left(\frac{x-y}{\varepsilon}\right) f(y)\, d\lambda(y) \\
&= -\frac{1}{\varepsilon^n} \int_\Omega \frac{\partial \varphi}{\partial y_i}\left(\frac{x-y}{\varepsilon}\right) f(y)\, d\lambda(y) \\
&= \frac{1}{\varepsilon^n} \int_\Omega \varphi\left(\frac{x-y}{\varepsilon}\right) \frac{\partial f(y)}{\partial y_i}\, d\lambda(y)
\end{aligned}$$

$$(2) \therefore \boxed{\frac{\partial f_\varepsilon}{\partial x_i}(x) = \left(\frac{\partial f}{\partial x_i}\right)_\varepsilon (x)}$$

Using now Theorem 335.1 we have:

$$\left\| \left(\frac{\partial f}{\partial x_i}\right)_\varepsilon - \frac{\partial f}{\partial x_i} \right\|_{p;\Omega'} \longrightarrow 0 \quad \text{as } \varepsilon \to 0.$$

(since $\frac{\partial f}{\partial x_i} \in L^p(\Omega)$).

Therefore, from (2):

$$(3) \quad \boxed{\left\| \frac{\partial f_\varepsilon}{\partial x_i} - \frac{\partial f}{\partial x_i} \right\|_{p;\Omega'} \longrightarrow 0}$$

Also, Theorem 335.1 gives:

$$(4) \quad \boxed{\| f_\varepsilon - f \|_{p;\Omega'} \longrightarrow 0}$$

From (3) and (4) we conclude:

$$\| f_\varepsilon - f \|_{1,p;\Omega'} = \| f_\varepsilon - f \|_{p;\Omega'} + \sum_{i=1}^{n} \left\| \frac{\partial f_\varepsilon}{\partial x_i} - \frac{\partial f}{\partial x_i} \right\|_{p;\Omega'}$$