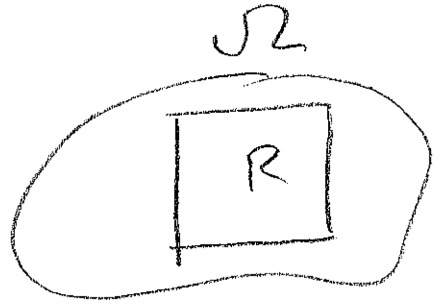
We now prove the converse:

Let $f \in L^p(R)$ and suppose that $\exists f^*$ s.t:

- $f^* = f$ λ_n-a.e.
- f^* is absolutely continuous on almost all line segments of R that are parallel to the coordinate axes.
- $\frac{\partial f^*}{\partial x_i} \in L^p(R)$.

Then: $f \in W^{1,p}(R)$

<u>Proof</u>:



$R = (a_1, b_1) \times \cdots \times (a_n, b_n)$

Let $\varphi \in C_c^\infty(R)$

We now prove that $f^* \varphi$ is also absolutely continuous on almost all line segments of R that are parallel to the coordinate axes:

Since φ is also absolutely continuous on lines:

$$\sum |b_I - a_I| < \delta \Rightarrow \sum | f^*(b_I)\varphi(b_I) - f^*(a_I)\varphi(a_I)|$$

$$= \sum | f^*(b_I)\varphi(b_I) - f^*(a_I)\varphi(b_I) + f^*(a_I)\varphi(b_I) - f^*(a_I)\varphi(a_I)|$$

$$\leq \sum |\varphi(b_I)|\,|f^*(b_I) - f^*(a_I)| + |f^*(a_I)|\,|\varphi(b_I) - \varphi(a_I)|$$

$$\leq C_1 \sum |f^*(b_I) - f^*(a_I)| + C_2 \sum |\varphi(b_I) - \varphi(a_I)|$$

$$\leq C_1 \cdot \frac{\varepsilon}{2C_1} + C_2 \cdot \frac{\varepsilon}{2C_1} = \varepsilon.$$

Thus, for $1 \leq i \leq n$, we can apply the Fundamental Theorem of Calculus to obtain:

$$\int_{a_i}^{b_i} \frac{\partial (f^*\varphi)}{\partial x_i}(x', t)\, dt = 0, \quad \lambda_{n-1}\text{-a.e. } x' \in R_i$$

and therefore,

$$\int_{a_i}^{b_i} f^*(x', t) \frac{\partial \varphi}{\partial x_i}(x', t)\, dt = -\int_{a_i}^{b_i} \frac{\partial f^*}{\partial x_i}(x', t)\,\varphi(x', t)\, dt$$

Note: Recall Problem 7.23.
Establish the integration by parts formula: if f and g are absolutely continuous on $[a,b]$, then:

$$\int_a^b f'g\, d\lambda = f(b)g(b) - f(a)g(a) - \int_a^b fg'\, d\lambda$$

Fubini's Theorem implies:

$$-\int_R f^* \frac{\partial \varphi}{\partial x_i}\, d\lambda = \int_R \frac{\partial f^*}{\partial x_i} \varphi \, d\lambda \quad , \forall \varphi \in C_c^\infty(R)$$

The distribution $\frac{\partial f}{\partial x_i}$ is defined as:

$$\frac{\partial f}{\partial x_i}(\varphi) = -\int_R f \frac{\partial \varphi}{\partial x_i}\, d\lambda \quad , \varphi \in C_c^\infty(R)$$

Hence:

$$(*) \quad \frac{\partial f}{\partial x_i}(\varphi) = \int_R \frac{\partial f^*}{\partial x_i} \varphi \, d\lambda \quad , \forall \varphi \in C_c^\infty(R)$$

; since $f = f^*$ a.e.

From $(*)$, it follows that the distribution $\frac{\partial f}{\partial x_i}$ is the function $\frac{\partial f^*}{\partial x_i} \in L^p(R)$.

$$\therefore \frac{\partial f}{\partial x_i} \in L^p(R)$$

$$\therefore f \in W^{1,p}(R)$$

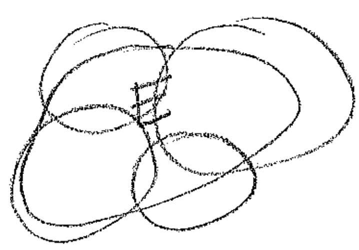
Smooth Partition of Unity.

Lemma 1: Let $\mathcal{G}$ be an open cover of a set $E \subset \mathbb{R}^n$. Then there exists a family $\mathcal{F}$ of functions $f \in C_c^\infty(\mathbb{R}^n)$ such that: $0 \le f \le 1$ and:

(i) For each $f \in \mathcal{F}$, there exists $U \in \mathcal{G}$ such that $\operatorname{spt} f \subset U$,

(ii) If $K \subset E$ is compact, then $\operatorname{spt} f \cap K \neq \emptyset$ for only finitely many $f \in \mathcal{F}$

(iii) $\displaystyle\sum_{f \in \mathcal{F}} f(x) = 1, \quad \forall x \in E$



$f \in \mathcal{F}$, $\operatorname{spt} f \subset U$

The family $\mathcal{F}$ is called a smooth partition of unity of E subordinate to the open covering $\mathcal{G}$.

Thm: Let $1 \le p < \infty$ and:

$$S := C^\infty(\Omega) \cap W^{1,p}(\Omega)$$

Then $\bar{S} = W^{1,p}(\Omega)$; that is, S is dense in $W^{1,p}(\Omega)$

Proof:

1.- Fix $\varepsilon > 0$ and define:

$$\Omega_K := \{x \in \Omega : d(x, \partial\Omega) > \tfrac{1}{K}\} \cap B(0, K) \qquad K = 1, 2, \ldots$$

$$\Omega_0 = \emptyset$$

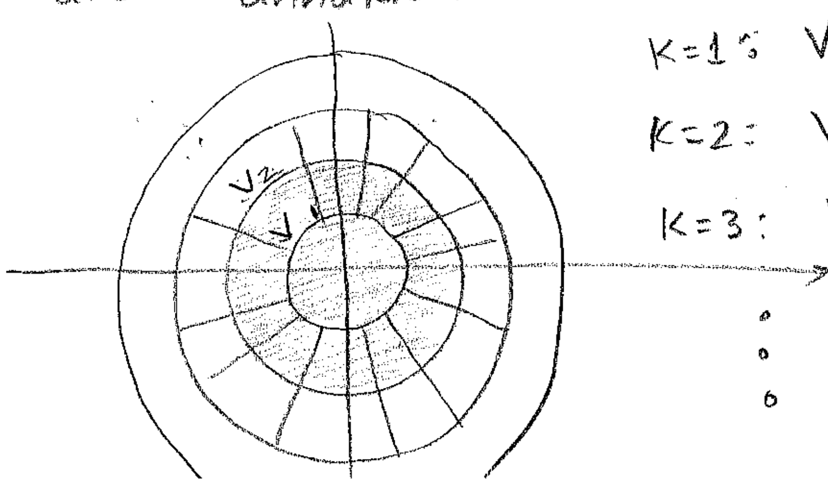
Then we have:

$$\Omega = \bigcup_{K=1}^{\infty} \Omega_K \qquad \bar{\Omega}_K \subset \Omega_{K+1}, \ \bar{\Omega}_K \text{ bounded.}$$

Set:

$$V_K := \Omega_{K+1} - \bar{\Omega}_{K-1}, \quad K = 1, 2, \ldots$$

For example, if $\Omega = \mathbb{R}^n$ then the $\{V_K\}$ are annulus:



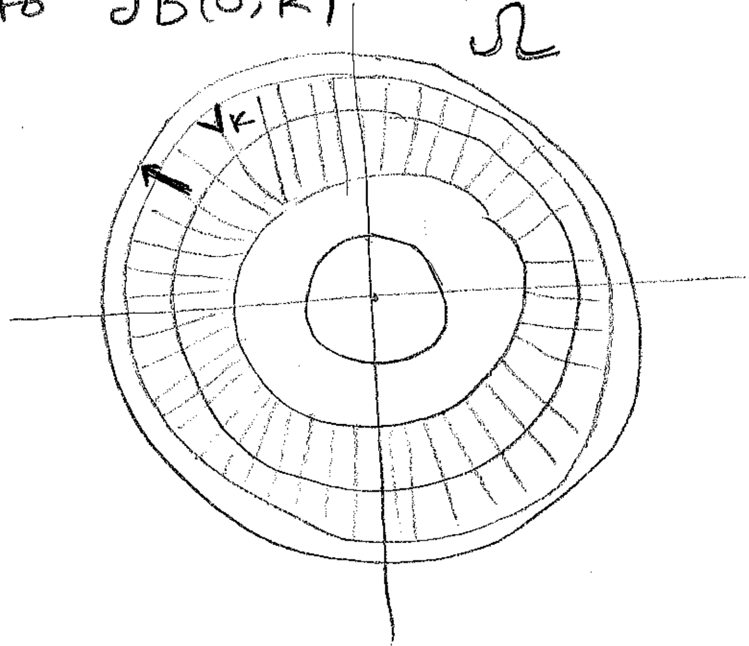
$K=1$: $V_1 = B(0,2) - \emptyset$

$K=2$: $V_2 = B(0,3) - \overline{B(0,1)}$

$K=3$: $V_3 = B(0,4) \setminus \overline{B(0,2)}$

$\vdots$

If $\Omega = B(0,R)$, the $\{V_K\}$ are annulus whose outer boundaries are closer and closer to $\partial B(0,R)$



Lemma 2: There exists $\{f_K\}_{K=1}^{\infty}$, a sequence of smooth functions such that

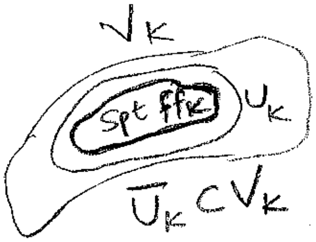
$$\boxed{\begin{array}{l} f_K \in C_c^{\infty}(V_K), \quad 0 \leq f_K \leq 1, \quad K=1,2,\ldots \\ \sum_{K=1}^{\infty} f_K \equiv 1 \quad \text{on } \Omega \end{array}}$$

Note:

$$f f_K \in W^{1,p}(\Omega), \quad K=1,2,\ldots$$
$$\text{spt}\,(f f_K) \subset V_K$$

We have seen the functions in $W^{1,p}$ can be approximated by Convolutions. Therefore, $\exists\, \varepsilon_k$ such that:

$$\operatorname{spt}\left(\varphi_{\varepsilon_k} * (ff_k)\right) \subset V_k$$



U_k, V_k open

$$\left(\int_\Omega |\varphi_{\varepsilon_k} * (ff_k) - ff_k|^p \, d\lambda(x)\right)^{1/p} < \frac{\varepsilon}{2^k}$$

$$\left(\int_\Omega |\varphi_{\varepsilon_k} * (\nabla(ff_k)) - \nabla(ff_k)|^p \, d\lambda(x)\right)^{1/p} < \frac{\varepsilon}{2^k}$$

Define:

$$\boxed{f_\varepsilon := \sum_{k=1}^{\infty} \varphi_{\varepsilon_k} * (ff_k)}$$

In some neighborhood of each $x \in \Omega$, there are only finitely many nonzero terms in this sum; hence

$$\boxed{f_\varepsilon \in C^\infty(\Omega)}$$

2.- For $x \in \Omega_K$, we have.

$$f(x) = \sum_{j=1}^{K} f_j(x) f(x)$$

and

$$f_\varepsilon(x) = \sum_{j=1}^{K} (\varphi_{\varepsilon_j} * ff_j)(x)$$

Hence:

$$\| f - f_\varepsilon \|_{1,p;\Omega_K} = \| \sum_{j=1}^{K} f_j f - \sum_{j=1}^{K} (ff_j)_{\varepsilon_j} \|_{1,p;\Omega_K}$$

$$\leq \sum_{j=1}^{K} \| f_j f - (ff_j)_{\varepsilon_j} \|_{1,p;\Omega_K}$$

$$\leq \sum_{j=1}^{\infty} \| f_j f - (ff_j)_{\varepsilon_j} \|_{1,p;\Omega}$$

$$\leq \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon$$

$\therefore \; \|f - f_\varepsilon\|_{1,p;\Omega_K} \leq \varepsilon \quad , \quad K = 1, 2, \ldots$

Applying the monotone convergence Theorem to let $K \to \infty$ we conclude

$$\boxed{\|f - f_\varepsilon\|_{1,p;\Omega} < \varepsilon , \quad f_\varepsilon \in C^\infty(\Omega) \cap W^{1,p}(\Omega)}$$