Theorem: Let $1 \leq p < n$ and let $\Omega \subset \mathbb{R}^n$ be a bounded open set. There is a constant $C = C(n,p)$ such that for $f \in W_0^{1,p}(\Omega)$,

$$\|f\|_{p^*;\Omega} \leq C \|\nabla f\|_{p;\Omega}$$

Note: $p^* = \frac{np}{n-p}$

Proof:

Step 1: Assume first that $p = 1$ and $f \in C_c^\infty(\mathbb{R}^n)$.

Using the Fundamental Theorem of Calculus and using the fact that f has compact support, it follows for each integer i, $1 \leq i \leq n$, that:

$$f(x_1, \dots, x_i, \dots, x_n) = \int_{-\infty}^{x_i} \frac{\partial f}{\partial x_i}(x_1, \dots, t_i, \dots, x_n)\, dt_i$$

$$\Rightarrow |f(x)| \leq \int_{-\infty}^{x_i} \left| \frac{\partial f}{\partial x_i}(x_1, \dots, t_i, \dots, x_n) \right| dt_i$$

$$\leq \int_{-\infty}^{\infty} \left| \frac{\partial f}{\partial x_i}(x_1, \dots, t_i, \dots, x_n) \right| dt_i$$

$$|f(x)| \le \int_{-\infty}^{\infty} |\nabla f(x_1,\dots,t_i,\dots,x_n)|\,dt_i, \qquad 1\le i\le n$$

$$\text{(A)} \quad \boxed{|f(x)|^{\frac{1}{n-1}} \le \left(\int_{-\infty}^{\infty} |\nabla f(x_1,\dots,t_i,\dots,x_n)|\,dt_i\right)^{\frac{1}{n-1}}, \quad i=1,2,\dots,n}$$

Since (A) is true for every i, we can multiply n times (A), one for each i to obtain:

$$|f(x)|^{\frac{n}{n-1}} \le \prod_{i=1}^{n}\left(\int_{-\infty}^{\infty} |\nabla f(x_1,\dots,t_i,\dots,x_n)|\,dt_i\right)^{\frac{1}{n-1}}$$

This can be rewritten as:

$$\text{(B)} \quad \boxed{|f(x)|^{\frac{n}{n-1}} \le \underbrace{\left(\int_{-\infty}^{\infty} |\nabla f(x)|\,dt_1\right)}_{(1)}^{\frac{1}{n-1}} \cdot \prod_{i=2}^{n}\left(\int_{-\infty}^{\infty} |\nabla f(x_1,\dots,t_i,\dots,x_n)|\,dt_i\right)^{\frac{1}{n-1}}}$$

Only the first factor on the right (1) is independent of x_1. We now integrate (B) with respect to x_1 and use the generalized Hölder inequality to obtain:

General Hölder inequality: Let $1 \le p_1, \dots, p_n \le \infty$ with $\frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_m} = 1$, and assume $u_K \in L^{p_K}(\Omega)$ for $K = 1, \dots, m$. Then:

$$\int_\Omega |u_1 \cdots u_m| \, dx \le \prod_{K=1}^{m} \|u_i\|_{p_k;\Omega}.$$

$$\int_{-\infty}^{\infty} |f(x)|^{\frac{n}{n-1}} dx_1 \le \left(\int_{-\infty}^{\infty} |\nabla f(x)| dt_1\right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} |\nabla f(x)| dt_i\right)^{\frac{1}{n-1}} dx_1$$

$$= \left(\int_{-\infty}^{\infty} |\nabla f(x)| dt_1\right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} |\nabla f(x)| dt_2\right]^{\frac{1}{n-1}} \left[\int_{-\infty}^{\infty} |\nabla f(x)| dt_3\right]^{\frac{1}{n-1}} \cdots \left[\int_{-\infty}^{\infty} |\nabla f(x)| dt_n\right]^{\frac{1}{n-1}} dx_1$$

$p_2 = n-1, \; p_3 = n-1, \dots, p_n = n-1$

$\frac{1}{p_2} = \frac{1}{n-1} \quad \frac{1}{p_3} = \frac{1}{n-1} \quad \frac{1}{p_n} = \frac{1}{n-1}$

$$\le \left(\int_{-\infty}^{\infty} |\nabla f(x)| dt_1\right)^{\frac{1}{n-1}} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\nabla f(x)| dt_2 \, dx_1\right)^{\frac{1}{n-1}} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\nabla f(x)| dt_3 \, dx_1\right)^{\frac{1}{n-1}} \cdots$$

$$\cdots \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\nabla f(x)| dt_n \, dx_1\right)^{\frac{1}{n-1}}$$

$$= \left(\int_{-\infty}^{\infty} |\nabla f(x)| dt_1\right)^{\frac{1}{n-1}} \left(\prod_{i=2}^{n} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\nabla f(x)| dt_i \, dx_1\right)^{\frac{1}{n-1}}$$

We have computed:

$$(C)\quad \boxed{\int_{-\infty}^{\infty}|f(x)|^{\frac{n}{n-1}}dx_1 \le \left(\int_{-\infty}^{\infty}|\nabla f(x)|dt_1\right)^{\frac{1}{n-1}}\left(\prod_{i=2}^{n}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\nabla f(x)|dx_i\,dt_i\right)^{\frac{1}{n-1}}}$$

We now integrate (C) with respect to x_2:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f|^{\frac{n}{n-1}}dx_1dx_2 \le$$

$$\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\nabla f(x)|\,dx_1\,dt_2\right)^{\frac{1}{n-1}}\int_{-\infty}^{\infty}\prod_{\substack{i=1\\ i\neq 2}}^{n} I_i^{\frac{1}{n-1}}\,dx_2,$$

$$I_1 := \int_{-\infty}^{\infty}|\nabla f|\,dt_1 \qquad I_i := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\nabla f|\,dx_1\,dt_i, \qquad i=3,\dots,n$$

Applying once more the generalized Hölder inequality, we find:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f|^{\frac{n}{n-1}}dx_1\,dx_2$$

$$\le \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\nabla f(x)|dx_1dt_2\right)^{\frac{1}{n-1}}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\nabla f|\,dt_1dx_2\right)^{\frac{1}{n-1}}\prod_{i=3}^{n}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\nabla f|\,dx_1dx_2\,dt_i\right)^{\frac{1}{n-1}}$$

Continuing this way for the remaining $n-2$ steps, we finally arrive at:

$$\int_{\mathbb{R}^n} |f|^{\frac{n}{n-1}} dx \leq \prod_{i=1}^{n} \left(\int_{\mathbb{R}^n} |\nabla f| \, dx_1 \ldots dt_i \ldots dx_n \right)^{\frac{1}{n-1}}$$

$$\therefore \left(\int_{\mathbb{R}^n} |f|^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} \leq \prod_{i=1}^{n} \left(\int_{\mathbb{R}^n} |\nabla f| \, dx \right)^{\frac{1}{n}}$$

$$= \int_{\mathbb{R}^n} |\nabla f| \, dx$$

$$\therefore \left(\int_{\mathbb{R}^n} |f|^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} \leq \int_{\mathbb{R}^n} |\nabla f(x)| \, dx$$

$$\therefore \quad (D) \quad \boxed{\|f\|_{\frac{n}{n-1}} \leq \int_{\mathbb{R}^n} |\nabla f| \, dx}, \quad f \in C_c^\infty(\mathbb{R}^n)$$

which is the desired result for $p=1$

Let $f \in C_c^\infty(\mathbb{R}^n)$.

Let $g := |f|^q$, ($q \geq 1$ will be determined later).

We have proved (D) for $f \in C_c^\infty(\mathbb{R}^n)$. However, a close examination of the proof reveals that we only need f to be an absolutely continuous function in each variable separately. Applying (D) to $g = |f|^q$ we obtain:

$$\|f^q\|_{\frac{n}{n-1}} \leq \int_{\mathbb{R}^n} |\nabla |f|^q| \, dx$$

$$= \int_{\mathbb{R}^n} q \, |f|^{q-1} \, |\nabla f| \, dx$$

$$\leq q \, \|f^{q-1}\|_{p'} \, \|\nabla f\|_p$$

$$\frac{1}{p'} + \frac{1}{p} = 1 \qquad \frac{1}{p'} = 1 - \frac{1}{p} = \frac{p-1}{p} \implies p' = \frac{p}{p-1}$$

We want q s.t: $(q-1)p' = (q-1) \cdot \frac{p}{p-1} = q \cdot \frac{n}{n-1}$

$$\Rightarrow q \cdot \frac{p}{p-1} - \frac{p}{p-1} = \frac{qn}{n-1}$$

$$\Rightarrow q\left(\frac{p}{p-1} - \frac{n}{n-1}\right) = \frac{p}{p-1} \implies q\left(\frac{pn - p - np + n}{(p-1)(n-1)}\right) = \frac{p}{p-1}$$

$$q\,\frac{(n-p)}{n-1} = p$$

$$\boxed{q = \frac{(n-1)p}{n-p}}$$

With this q:

- $q \cdot \dfrac{n}{n-1} = \dfrac{\cancel{(n-1)}p}{n-p} \cdot \dfrac{n}{\cancel{n-1}} = \dfrac{pn}{n-p}$

$$\boxed{\text{Hence: } \left(\int_{\mathbb{R}^n} |f|^{\frac{np}{n-p}}\right)^{\frac{n-1}{n}} \le q\left(\int_{\mathbb{R}^n} |f|^{\frac{np}{n-p}}\right)^{1/p'} \|\nabla f\|_p}$$

Now:

$$\frac{n-1}{n} - \frac{1}{p'} = \frac{n-1}{n} - \left(1 - \frac{1}{p}\right) = \frac{n-1}{n} - 1 + \frac{1}{p}$$

$$= \frac{\cancel{pn} - p - \cancel{np} + n}{np} = \frac{n-p}{np}$$

$$\Rightarrow \left(\int_{\mathbb{R}^n} |f|^{\frac{np}{n-p}}\right)^{\frac{n-p}{np}} \le q\, \|\nabla f\|_{p;\mathbb{R}^n}$$

$$\text{(E)}\quad \boxed{\therefore\ \|f\|_{\frac{np}{n-p};\mathbb{R}^n} \le \frac{(n-1)p}{n-p}\, \|\nabla f\|_{p;\mathbb{R}^n}}\ ,\ f \in C_c^\infty(\mathbb{R}^n)$$

Step 2: Let Ω be a bounded open set and let $f \in W_0^{1,p}(\Omega)$.
Let $\{f_i\}$ be a sequence of functions in $C_c^\infty(\Omega)$ converging to f in the Sobolev norm; i.e:

$$(F) \quad \boxed{\|f_i - f_j\|_{1,p;\Omega} \longrightarrow 0}$$

Applying (E) to $\|f_i - f_j\|$:

$$\|f_i - f_j\|_{p^*;\mathbb{R}^n} \le C \|f_i - f_j\|_{1,p;\mathbb{R}^n},$$

where we have extended the functions f_i by zero outside Ω. Therefore we have:

$$(G) \quad \boxed{\|f_i - f_j\|_{p^*;\Omega} \le C \|f_i - f_j\|_{1,p;\Omega}}$$

From (F) and (G) it follows that $\{f_i\}$ is Cauchy in $L^{p^*}(\Omega)$, and hence $\exists g \in L^{p^*}(\Omega)$ such that

$$\boxed{f_i \to g \text{ in } L^{p^*}(\Omega)}$$

Since $p^* \geq p$ and Ω bounded we have:

$$f_i \to g \quad \text{in } L^p(\Omega),$$

but we know that $f_i \to f$ in $L^p(\Omega)$, and hence $f = g$. That is:

$$\text{(H)} \quad \boxed{f_i \to f \quad \text{in } L^{p^*}(\Omega)}$$

Since $|\nabla f_i| \to |\nabla f|$ in $L^p(\Omega)$, we can let $i \to \infty$ and use (H) in:

$$\|f_i\|_{p^*;\Omega} \leq C \|\nabla f_i\|_{p;\Omega}$$

to obtain:

$$\|f\|_{p^*;\Omega} \leq C \|\nabla f\|_{p;\Omega}.$$