Proof of Theorem 1:

Let

$$m := \inf \left\{ \int_\Omega |\nabla f|^2 \, dx : f - \psi \in W_0^{1,2}(\Omega) \right\}$$

Note: Since $\psi \in W^{1,2}(\Omega)$, this definition requires $f \in W^{1,2}(\Omega)$ and therefore:

$$|\nabla f| \in L^2(\Omega)$$

Let $\{f_i\}$ be a sequence in $W^{1,2}(\Omega)$ such that $f_i - \psi \in W_0^{1,2}(\Omega)$ and

$$\text{(A)} \quad \boxed{\int_\Omega |\nabla f_i|^2 \, dx \to m \quad \text{as} \quad i \to \infty}$$

We have:

$$\|f_i - \psi\|_{2;\Omega} = \left(\int_\Omega |f_i - \psi|^2 dx \right)^{1/2}$$

$$\leq \left(\int_\Omega 1^p \right)^{\frac{1}{2p}} \left(\int_\Omega |f_i - \psi|^{2p'} d\lambda(x) \right)^{\frac{1}{2p'}}$$

We want

$$2p' = 2^* = \frac{2n}{n-2}$$

$$\therefore \quad p' = \frac{n}{n-2}$$

$$\frac{1}{p} + \frac{1}{p'} = 1 \quad \Rightarrow \quad \frac{1}{p} = 1 - \frac{1}{p'}$$

$$= 1 - \frac{n-2}{n}$$

$$= \frac{n-n+2}{n} = \frac{2}{n}$$

$$\therefore \quad p' = \frac{n}{n-2} \quad \& \quad p = \frac{n}{2}$$

$\therefore$

$$\| f_i - \psi \|_{2;\Omega} \leq \lambda(\Omega)^{\frac{1}{n}} \left(\int_\Omega |f_i - \psi|^{p^*} \right)^{\frac{1}{p^*}}, \quad \frac{1}{n} = \frac{1}{2} - \frac{1}{2^*}$$

$$= \lambda(\Omega)^{\frac{1}{2} - \frac{1}{2^*}} \| f_i - \psi \|_{p^*;\Omega} \qquad = \frac{1}{2} - \frac{n-2}{2n}$$

$$= \frac{\cancel{n} - \cancel{n} + 2}{2n} = \frac{1}{n}$$

$$\boxed{\therefore \quad \| f_i - \psi \|_{2;\Omega} \leq \lambda(\Omega)^{\frac{1}{2} - \frac{1}{2^*}} \| f_i - \psi \|_{p^*;\Omega}} \quad (B)$$

Using the Sobolev Imbedding Theorem:

$$\| f_i - \psi \|_{p^*;\Omega} \le \| \nabla (f_i - \psi) \|_{2;\Omega}$$

we obtain from (B):

$$\boxed{\| f_i - \psi \|_{2;\Omega} \le C \, \| \nabla (f_i - \psi) \|_{2;\Omega}} \quad (C)$$

From (A) and (C):

$$\boxed{\begin{aligned} \| f_i \|_{2;\Omega} &\le C \quad , \; i = 1, 2, \ldots \\ \| \nabla f_i \|_{2;\Omega} &\le C \quad , \; i = 1, 2, \ldots \end{aligned}}$$

Since $L^2(\Omega)$ is reflexive, there exists $f \in L^2(\Omega)$ and $\vec{g} = (g_1, \ldots, g_n) \in L^2(\Omega)$ and a subsequence of $\{f_i\}$, denoted by the full sequence, such that:

$$\boxed{\begin{aligned} f_i &\rightharpoonup f \quad \text{weakly in } L^2(\Omega) \\ \nabla f_i &\rightharpoonup \vec{g} \quad \text{weakly in } L^2(\Omega) \end{aligned}}$$

11.291

Claim: $f \in W^{1,2}(\Omega)$ and

$\nabla f = \vec{g}$.

Fix $k \in \{1, \dots, n\}$. We need to check that $\frac{\partial f}{\partial x_k} = g_k$. Let $\varphi \in C_c^\infty(\Omega)$. Then:

$$\left\langle \frac{\partial f}{\partial x_k}, \varphi \right\rangle = -\left\langle f, \frac{\partial \varphi}{\partial x_k} \right\rangle$$

$$= -\int_\Omega f \frac{\partial \varphi}{\partial x_k} \, d\lambda(x)$$

$$= -\lim_{i \to \infty} \int_\Omega f_i \frac{\partial \varphi}{\partial x_k} \, d\lambda(x), \quad \text{since } f_i \to f \text{ in } L^2(\Omega)$$

$$= \lim_{i \to \infty} \int_\Omega \frac{\partial f_i}{\partial x_k} \varphi \, d\lambda(x), \quad \text{since } f_i \in W^{1,2}(\Omega)$$

$$= \int_\Omega g_k \varphi \, d\lambda(x)$$

$\therefore \frac{\partial f}{\partial x_k} = g_k \in L^2(\Omega)$, $k = 1, 2, \dots, n$

$\therefore f \in W^{1,2}(\Omega)$ and $\nabla f = \vec{g}$.

We have then

$$\boxed{\begin{aligned} f_i &\rightharpoonup f \quad \text{in } L^2(\Omega) \\ \nabla f_i &\rightharpoonup \nabla f \quad \text{in } L^2(\Omega) \end{aligned}} \quad (D)$$

<u>Claim:</u> (D) is the same that:

$$f_i \rightharpoonup f \quad \text{weakly in } W^{1,2}(\Omega)$$

<u>Claim</u>: $f - \psi \in W_0^{1,2}(\Omega)$

From (A) and (C)

$$\|f_i - \psi\|_{2;\Omega} \leq C \;, \quad f_i - \psi \in W_0^{1,2}(\Omega)$$

$$\|\nabla(f_i - \psi)\|_{2;\Omega} \leq C \;, \quad i = 1, 2, \ldots$$

Hence

$$\|f_i - \psi\|_{1,p;\Omega} < C \qquad i = 1, 2, \ldots$$

Since $W^{1,2}(\Omega)$ is reflexive, there exists a subsequence (denoted by the full sequence) and $h \in W^{1,2}(\Omega)$ such that

$$(E) \quad \boxed{f_i - \psi \longrightarrow h \quad \text{weakly in } W^{1,2}(\Omega)}$$

From (D):

(F) $\boxed{f_i - \Psi \rightharpoonup f - \Psi \text{ weakly in } W^{1,2}(\Omega)}$

(E), (F) and the uniqueness of weak limit yield:

$$h = f - \Psi$$

From Theorem 290.1, it follows that $h \in \overline{W}$, (in the strong Sobolev norm), where W is the span of the sequence $\{f_i - \Psi\} \in W_0^{1,2}$. Hence, $\exists\, h_i \in W_0^{1,2}(\Omega)$ such that $h_i \to h$ strongly in $W^{1,2}(\Omega)$.

We recall the continuous trace operator:

$$Tr : W^{1,2}(\Omega) \to L^2(\partial\Omega)$$
$$\|Tr\, u\|_{2;\partial\Omega} \leq C \|u\|_{1,2;\Omega}$$

Hence, $Tr(h_i) \to T(h)$ and since $Tr(h_i) \equiv 0$ then $T(h) \equiv 0$. Thus $h = f - \Psi \in W_0^{1,2}(\Omega)$ and f is admissible.

(11.294)

Claim: The infimum m is attained at f.

We have seen that f is an admissible function; i.e. $f - \Psi \in W_0^{1,2}(\Omega)$

Since $\nabla f_i \rightharpoonup \nabla f$ weakly in $L^2(\Omega)$, we can use the lower semicontinuity in Theorem 290.1 to obtain:

$$\int_\Omega |\nabla f|^2 \, d\lambda(x) \leq \liminf_{i \to \infty} \int_\Omega |\nabla f_i|^2 \, d\lambda(x)$$

$$= m.$$

(Modify proof in Theorem 290.1 to get $\|x\|^2 \leq \liminf_{i \to \infty} \|x_k\|^i$

Hence:

$$\boxed{\int_\Omega |\nabla f|^2 \, d\lambda(x) = m}$$

<u>Claim:</u> f is weakly harmonic.

Let $\varphi \in C_c^\infty(\Omega)$. Define:

$$\alpha(t) = \int_\Omega |\nabla(f+t\varphi)|^2 \, d\lambda(x)$$

$$= \int_\Omega |\nabla f|^2 + 2t\, \nabla f \cdot \nabla \varphi + t^2 |\nabla \varphi|^2 \, d\lambda(x)$$

Clearly α has a local minimum at $t=0$. By exercise 6.15, we can differentiate under the integral to get:

$$\alpha'(t) = \int_\Omega 0 + 2\nabla f \cdot \nabla \varphi + 2t|\nabla \varphi|^2 \, d\lambda(x)$$

$$\therefore \quad \alpha'(0) = 0 = 2\int_\Omega \nabla f \cdot \nabla \varphi \, d\lambda(x)$$

$$\therefore \quad \int_\Omega \nabla f \cdot \nabla \varphi \, d\lambda(x) = 0 \quad \forall \varphi \in C_c^\infty(\Omega).$$

$\therefore$ f is weakly harmonic; i.e, f is a weak solution of the Laplace's equation.