Let $\Omega$ be a bounded open set and $\Upsilon \in W^{1,2}(\Omega)$.
We have shown that:
$\exists f \in W^{1,2}(\Omega)$, $f - \Upsilon \in W_0^{1,2}(\Omega)$ such that

$$\Delta f(\varphi) = \int_\Omega f \Delta \varphi = 0 \quad \forall \varphi \in C_c^\infty(\Omega);$$

i.e $f$ is a weakly harmonic function. In order to prove Theorem 2; i.e. that $f \in C^\infty(\Omega)$ we need some previous results:

<u>Theorem</u>: If $f \in W^{1,2}_{loc}(\Omega)$ is weakly harmonic, then $f$ is continuous in $\Omega$ and:

$$f(x_0) = \fint_{B(x_0,r)} f(y)\, d\lambda(y),$$

whenever $\overline{B}(x_0, r) \subset \Omega$.

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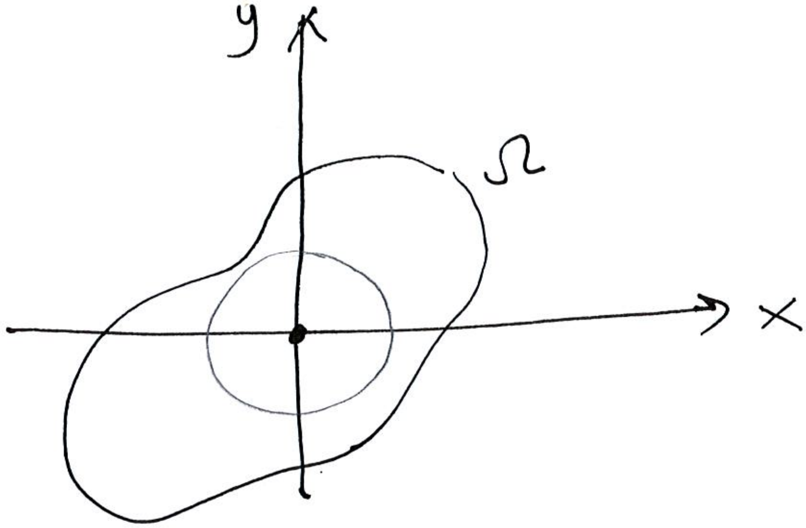
Proof: WLOG take $x_0=0$




Let $0<\delta< d(x_0,\partial\Omega)$. We will use the formula:

$$\int_{B(x_0,\delta)} f(x)\,d\lambda(x) = \int_0^\delta \int_{\partial B(0,1)} f(r,z)\, r^{n-1}\, d\mathcal{H}^{n-1}(z)\,dr.$$

Define:

$$F(r) = \int_{\partial B(0,1)} f(r,z)\, d\mathcal{H}^{n-1}(z), \quad 0<r<d(x_0,\partial\Omega)$$

Claim 1: $F(r)$ is constant for every $r$. that is, there exists a number $\alpha$ such that $F(r)=\alpha$, for all $0<r<d(x_0,\partial\Omega)$

From Claim 1 we compute:

$$\int_{B(x_0,\delta)} f(x)\, d\lambda(x) = \int_0^{\delta} \alpha r^{n-1}\, dr$$

$$= \alpha \frac{r^n}{n}\Big]_0^{\delta} = \alpha \frac{\delta^n}{n}$$

Hence:

$$\frac{1}{\delta^n} \int_{B(x_0,\delta)} f(x)\, d\lambda(x) = \frac{\alpha}{n}$$

and therefore:

$$\frac{1}{\lambda(B(x_0,\delta))} \int_{B(x_0,\delta)} f(x)\, d\lambda(x) = \alpha(x_0)\, c(n)$$

We have shown:

$$\boxed{\begin{array}{l} \displaystyle ⨍_{B(x_0,\delta)} f(y)\, d\lambda(y) = \alpha(x_0)\, c(n) \\ \text{for } 0 < \delta < d(x_0, \partial\Omega), \text{ \& every } x_0 \in \Omega \end{array}} \quad (A)$$

Since almost every $x_0 \in \Omega$ is a Lebesgue point for $f$ we have:

$$f(x_0) = \lim_{\delta \to 0} ⨍_{B(x_0,\delta)} f(y)\, dy, \qquad \lambda\text{-a.e. } x_0$$

Hence, by (A):

(B). $\boxed{\begin{array}{l} f(x_0) = ⨍_{B(x_0,r)} f(y)\, dy \\ \text{for } \lambda\text{-a.e. } x_0 \in \Omega \text{ and} \\ \text{any ball } \overline{B}(x_0,r) \subset \Omega \end{array}}$

<u>Claim</u>: $f$ can be redefined on a set of measure zero in such a way as to ensure its continuity in $\Omega$.

Indeed, if $x_0$ is not a Lebesgue point and $\overline{B}(x_0,r) \subset \Omega$, <u>define</u>:

$$\underset{\substack{\uparrow \\ \text{redefine}}}{f(x_0)} := ⨍_{B(x_0,r)} f(y)\, dy$$

$f$ is now continuous in $\Omega$. If $x_0 \in \Omega$ and $x_n \to x_0$, $x_n \in \Omega$, then we choose $r$ small enough so that

$$\bar{B}(x_n, r) \subset \Omega, \quad \bar{B}(x_0, r) \subset \Omega.$$

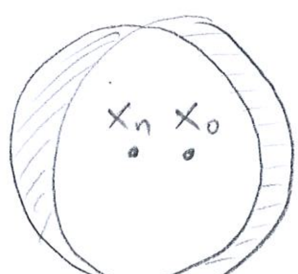
Thus:

$$|f(x_n) - f(x_0)| = \left| \int_{B(x_n,r)} f(y)\,dy - \int_{B(x_0,r)} f(y)\,dy \right|$$

$$\leq \int_{B(x_n,r) \Delta B(x_0,r)} |f(y)|\,dy$$

where:

$$B(x_n,r) \Delta B(x_0,r) = \left[B(x_n,r) \setminus B(x_0,r)\right] \cup \left[B(x_0,r) \setminus B(x_n,r)\right]$$




Given $\varepsilon > 0$, $\exists N$ s.t for $n \geq N$:

$\lambda\left(B(x_n,r) \Delta B(x_0,r)\right) < \delta$ and $\int_{B(x_n,r) \Delta B(x_0,r)} |f(y)|\,dy < \varepsilon$

$\therefore$ $|f(x_n) - f(x_0)| \leq \varepsilon$, $n \geq N$. Thus $f(x_n) \to f(x_0)$.

$\therefore$ $f$ is continuous in $\Omega$.

# Notes on Area and Coarea formula.

Let

$f: \mathbb{R}^n \to \mathbb{R}^m$, $m \geq n$, $f$ Lipschitz.

$A \subset \mathbb{R}^n$ Lebesgue measurable. Then:

(1) $$\boxed{\int_{\mathbb{R}^m} \mathcal{H}^0 (A \cap f^{-1}(y))\, d\mathcal{H}^n(y) = \int_A |Jf(x)|\, d\lambda(x)}$$

AREA FORMULA

For $m \geq n$, we can write:

$$df(x) = O \cdot S,$$

where $O$ is an $m \times n$ orthogonal matrix and $S$ is an $n \times n$ symmetric matrix. In (1) we define:

$$Jf(x) = \det S$$

If $f$ is 1-1 and $n = m$:

$$\int_{f(A)} d\lambda(y) = \int_A |Jf(x)|\, d\lambda(x)$$

and for any integrable function $g$:

(2) $$\boxed{\int_{f(A)} g(y)\, d\lambda(y) = \int_A g \circ f(x)\, |Jf(x)|\, d\lambda(x)}$$

<u>Ex 1</u>: Compute the volume inside the Ellipsoid:

$$E = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \right\}$$

Let

$$B = \left\{ (x, y, z) \in \mathbb{R}^3 : u^2 + v^2 + w^2 \leq 1 \right\}$$

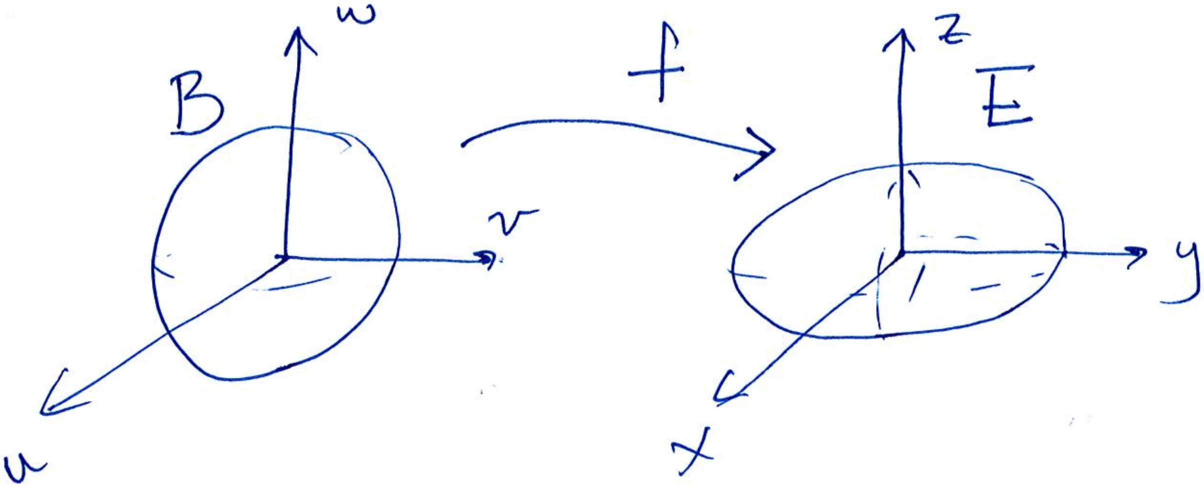
Let

$f: \mathbb{R}^3 \to \mathbb{R}^3$

$$f(u, v, w) = (au, bv, cw)$$

$$= (x(u,v,w), y(u,v,w), z(u,v,w))$$

<u>Claim</u>: $f(B) = E$ and $f|_B$ is 1-1



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Fix $0 < r \le 1$.

Let

$S_r = \{(u, v, w) : u^2 + v^2 + w^2 = r^2\}$ the

sphere of radius $r$.

Let $(x, y, z) \in f(S_r)$. Then:

$$x = au$$
$$y = bv$$
$$z = cw$$

implies:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = u^2 + v^2 + w^2 = r^2$$

$$\therefore \frac{x^2}{(ra)^2} + \frac{y^2}{(rb)^2} + \frac{z^2}{(rc)^2} = 1$$

which is an inner ellipsoid, say $E_r$. Thus:

$$f(S_r) = E_r.$$

Clearly, from here it follows

$$f(B) = E \qquad f|_B \ 1\text{-}1.$$

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Applying the Change of Variables formula:

$$\int_{T(B)} d\lambda(x,y,z) = \int_B |Jf(x)| \, d\lambda(u,v,w)$$

But:

$$Jf(x) = \det(df(x)),$$ where

$$df(x) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

$\therefore \; Jf(x) = abc.$

$$\therefore \int_E d\lambda(x,y,z) = \int_B abc \, d\lambda(u,v,w)$$

$$= abc \, \lambda(B) = \frac{4}{3}\pi abc.$$

$$\therefore \boxed{\lambda(E) = \text{Volume of } E = \frac{4}{3}\pi abc}$$