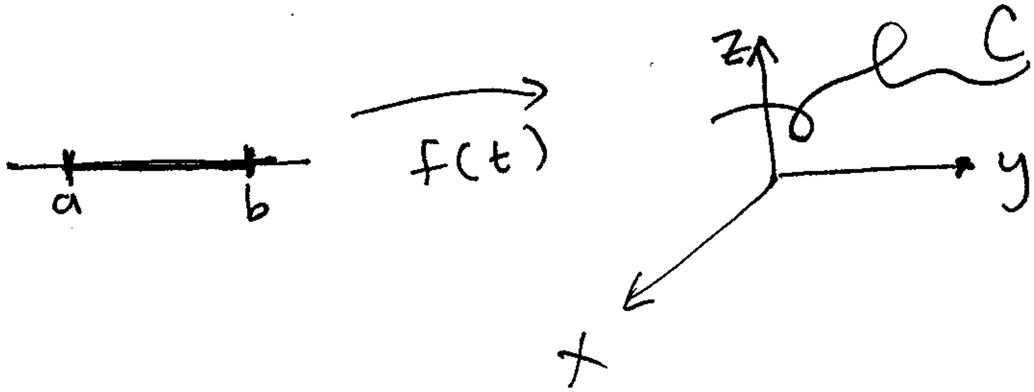
Ex 2. Length of a Curve

Let $f(t): \mathbb{R} \to \mathbb{R}^3$ Lipschitz and 1-1, Write:

$$f(t) = (x(t), y(t), z(t))$$

For $-\infty < a < b < \infty$ define the curve:

$$C := f([a,b]) \subset \mathbb{R}^3$$



From the area formula:

$$\int_{f([a,b])} d\mathcal{H}^1(y) = \int_a^b |Jf(t)|\, d\lambda(t)$$

$$\therefore \mathcal{H}^1(C) = \text{lenght of } C = \int_a^b |Jf(t)|\, d\lambda(t).$$

We have:

$$df(x) = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} \qquad \dot{x}(t) = \frac{dx}{dt}$$

The polar decomposition for the 3×1 matrix $df(t)$ gives:

$$df(t) = O\cdot S, \quad \text{where}$$

O is an 3×1 orthogonal matrix

S is a 1×1 matrix

$$\therefore \quad df(x) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \cdot \beta \quad , \quad \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$$

$$\therefore \quad \alpha_1\beta = \dot{x}(t)$$
$$\alpha_2\beta = \dot{y}(t)$$
$$\alpha_3\beta = \dot{z}(t)$$

$$\Rightarrow \alpha_1^2\beta^2 = (\dot{x}(t))^2, \quad \alpha_2^2\beta^2 = (\dot{y}(t))^2, \quad \alpha_3^2\beta^2 = (\dot{z}(t))^2$$

$$\Rightarrow \beta^2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2) = (\dot{x}(t))^2 + (\dot{y}(t))^2 + (\dot{z}(t))^2$$

$$\Rightarrow \beta = \sqrt{(\dot{x}(t))^2 + (\dot{y}(t))^2 + (\dot{z}(t))^2} = |f'(t)|$$

Thus,

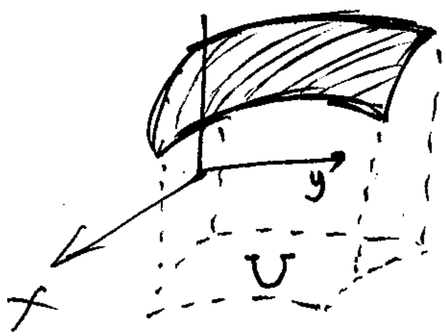
$$Jf(x) = \det S = |f'(t)|$$

We conclude:

$$\boxed{\mathcal{H}^1(C) = \int_a^b |f'(t)| \, d\lambda(t)}$$

Ex 3: Surface area of a graph

Let $g : \mathbb{R}^2 \to \mathbb{R}$ Lipschitz.



Define:

$$f : \mathbb{R}^2 \to \mathbb{R}^3$$

$$f(x,y) = (x, y, g(x,y)), \quad f \text{ is } 1\text{-}1 \text{ on } U$$

Then:

$$df(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{3\times 2}$$

The Binet-Cauchy Formula (Theorem 4, "Measure Theory and fine properties of functions) implies:

$$(Jf)^2 = \text{sum of squares of } (2\times 2)\text{-subdeterminants}$$

$$= 1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2$$

$\therefore$

$$|Jf(x,y)| = \sqrt{1+|\nabla g(x,y)|^2}$$

From Area Formula:

$$\int_{g(U)} d\mathcal{H}^2 = \int_U |Jf(x,y)|\, dx\, dy$$

$$= \int_U \sqrt{1+|\nabla g|^2}$$

$$\therefore \boxed{\mathcal{H}^2(U) = \text{Area of } U = \int_U \sqrt{1+|\nabla g|^2}}$$

Recall the Plateau Problem:

Minimize $\int_U \sqrt{1+|\nabla g|^2}$, over all function g such that $g = \Upsilon$ on ∂U.

If g minimizes area, then g satisfies the minimal surface equation:

$$\boxed{\operatorname{div}\left(\frac{\nabla g}{(1+|\nabla g|^2)}\right) = 0. \quad (*)}$$

Indeed, Let $\alpha(t) = \int_U \sqrt{1+|\nabla(g+t\varphi)|^2}\, dx\, dy$

$$\therefore \alpha'(t) = \int_U \frac{1}{2}\left(1+|\nabla(g+t\varphi)|^2\right)^{-1/2} \cdot \frac{d}{dt}\left(1+(g_x+t\varphi_x)^2 + (g_y+t\varphi_y)^2\right)$$

$$\therefore \alpha'(t) = \int_U \frac{1}{2}\left(1+|\nabla(g+t\varphi)|^2\right)^{-1/2}\left[2(g_x+t\varphi_x)\varphi_x + 2(g_y+t\varphi_y)\varphi_y\right]$$

$$\Rightarrow \alpha'(0) = \int_U (1+|\nabla g|^2)^{-1/2}(\nabla g \cdot \nabla \varphi)\, dx\, dy \Rightarrow \int_U \operatorname{div}\left(\frac{\nabla g}{\sqrt{1+|\nabla g|^2}}\right)\varphi = 0 \quad \forall \varphi$$

$$\Rightarrow (*)$$

The Coarea formula

Let $f: \mathbb{R}^n \to \mathbb{R}^m$ Lipschitz, $n \geq m$. Then, for each Lebesgue measurable set $A \subset \mathbb{R}^n$,

(1) $$\boxed{\int_A |Jf(x)|\,dx = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(y))\,dy}$$

COAREA FORMULA

Polar decomposition: The $m\times n$ matrix $df(x)$ can be written as:

$$df(x) = S \cdot O^*$$

where:

S in $m\times m$ symmetric matrix

O^* is the $m\times n$ **adjoint** matrix of the orthogonal matrix O

In (1) we define:

$$Jf(x) = \det S.$$

If $f: \mathbb{R}^n \to \mathbb{R}^m$, $n \geq m$ is Lipschitz and $g \in L^1(\mathbb{R}^n)$

Then $g|_{f^{-1}(y)}$ is $\mathcal{H}^{n-m}$- integrable for λ_m-a.e. y

and:

(2) $$\boxed{\int_{\mathbb{R}^n} g(x)\,|Jf(x)|\,dx = \int_{\mathbb{R}^m}\left[\int_{f^{-1}(y)} g\,d\mathcal{H}^{n-m}\right]dy}$$

Ex. 1: Let $g : \mathbb{R}^n \to \mathbb{R}$ be integrable. Then:

$$\int_{\mathbb{R}^n} g(x)\, d\lambda(x) = \int_0^\infty \int_{\partial B(0,r)} g(r,y)\, d\mathcal{H}^{n-1}(y)\, dr$$

In particular, we see:

$$\frac{d}{dr}\left(\int_{B(0,r)} g\, d\lambda(x)\right) = \int_{\partial B(0,r)} g\, d\mathcal{H}^{n-1}$$

for λ_1-a.e. $r > 0$.

Proof: Set $f(x) = |x| = (x_1^2 + \dots + x_n^2)^{1/2}$

$$\frac{\partial f}{\partial x_1} = \frac{1}{2}(x_1^2 + \dots + x_n^2)^{-1/2}(2x_1) = \frac{x_1}{|x|}$$

$$\frac{\partial f}{\partial x_2} = \frac{1}{2}(x_1^2 + \dots + x_n^2)^{-1/2}(2x_2) = \frac{x_2}{|x|}$$

$$\vdots$$

$$\frac{\partial f}{\partial x_n} = \frac{1}{2}(x_1^2 + \dots + x_n^2)^{-1/2}(2x_n) = \frac{x_n}{|x|}$$

$$df(x) = \left(\frac{x_1}{|x|}, \dots, \frac{x_n}{|x|}\right) = \beta \cdot (\alpha_1, \dots, \alpha_n)$$

$$\alpha_1 \beta = \frac{x_1}{|x|}, \quad \dots, \quad \alpha_n \beta = \frac{x_n}{|x|}$$

$$\beta^2(\alpha_1^2 + \dots + \alpha_n^2) = \frac{x_1^2}{|x|^2} + \dots + \frac{x_n^2}{|x|^2} = 1 \Rightarrow \beta = 1$$

$\therefore \; |Jf(x)| = 1$

We apply now the Coarea formula:

$$\int_{\mathbb{R}^n} g(x)\, |Jf(x)|\, d\lambda(x) = \int_0^\infty \left[\int_{f^{-1}(r)} g \, d\mathcal{H}^{n-1} \right] dr$$

with $f(x) = |x|$, $f^{-1}(r) = \partial B(0,r)$. Thus,

since $|Jf(x)| = 1$, we conclude:

$$\int_{\mathbb{R}^n} g(x)\, d\lambda(x) = \int_0^\infty \left(\int_{\partial B(0,r)} g \, d\mathcal{H}^{n-1} \right) dr.$$

In particular, for $r > 0$.

$$\int_{B(0,r)} g \, d\lambda(x) = \int_0^r \underbrace{\int_{\partial B(0,t)} g(t,y)\, d\mathcal{H}^{n-1}(y)}_{f(t)}\, dr$$

$$= \int_0^r f(t)\, dt.$$

The fundamental theorem of Calculus gives:

$$\frac{d}{dr} \int_{B(0,r)} g \, d\lambda(x) = \frac{d}{dr} \int_0^r f(t)\, dt$$

$$= f(r) \quad , \lambda_1\text{-a.e. } t$$

$$= \int_{\partial B(0,r)} g \, d\mathcal{H}^{n-1} \quad , \lambda_1\text{-a.e. } r. \qquad \blacksquare$$

Ex. 2: Let $f : \mathbb{R}^n \to \mathbb{R}$ Lebesgue integrable. We have seen in Ex. 1 that, for any $\delta > 0$:

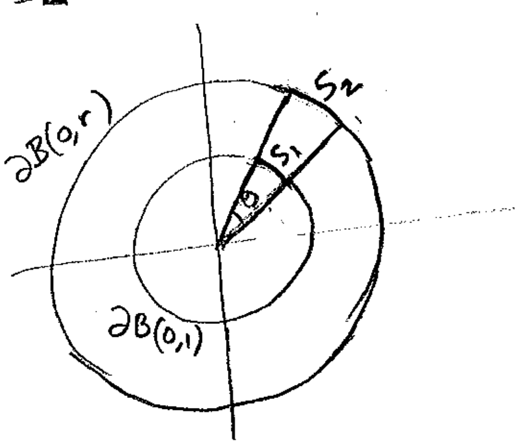
$$\int_{B(0,\delta)} f(x)\, d\lambda(x) = \int_0^\delta \int_{\partial B(0,r)} f(r,y)\, d\mathcal{H}^{n-1}(y)\, dr$$

We can also write this as:

$$\int_{B(0,\delta)} f(x)\, d\lambda(x) = \int_0^\delta \int_{\partial B(0,1)} f(r,y)\, r^{n-1}\, d\mathcal{H}^{n-1}(y)\, dr$$

This is because the area of $\partial B(0,r)$ is r^{n-1} times the area of $\partial B(0,1)$:

For $n=2$

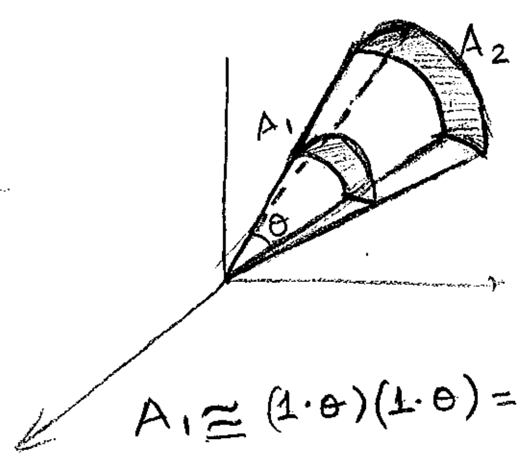


$S_1 = 1 \cdot \theta = \theta$

$S_2 = r\theta = r S_1$

$\therefore S_2 = r^{n-1} S_1$

For $n=3$



$A_1 \cong (1\cdot\theta)(1\cdot\theta) = \theta^2$

$A_2 \cong (r\theta)(r\theta) = r^2\theta^2$

$\therefore A_2 = r^2 A_1$

$\therefore A_2 = r^{n-1} A_1$

Ex 3: Assume $f: \mathbb{R}^n \to \mathbb{R}$ is Lipschitz. Then:

$$\int_{\mathbb{R}^n} |\nabla f| \, d\lambda(x) = \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(\{f = t\}) \, dt$$

Proof:

We have

$$df(x) = \left(\frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_n} \right) = \nabla f(x)$$

We write

$$df(x) = \beta \cdot (\alpha_1 \cdots \alpha_n)$$

$$\therefore \quad \frac{\partial f}{\partial x_1} = \beta\alpha_1, \quad \frac{\partial f}{\partial x_2} = \beta\alpha_2, \quad \ldots, \quad \frac{\partial f}{\partial x_n} = \beta\alpha_n$$

$$\therefore \quad \beta^2(\alpha_1^2 + \cdots + \alpha_n^2) = \left(\frac{\partial f}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial f}{\partial x_n}\right)^2$$

$$\therefore \quad \beta = \sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial f}{\partial x_n}\right)^2}, \quad \text{since } \alpha_1^2 + \cdots + \alpha_n^2 = 1$$

Hence $|Jf(x)| = |\nabla f(x)|$.

Applying the Coarea formula:

$$\int_{\mathbb{R}^n} |Jf(x)| \, d\lambda(x) = \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(f^{-1}(t)) \, dt$$

Thus: $$\int_{\mathbb{R}^n} |\nabla f| \, d\lambda(x) = \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(f^{-1}(t)) \, dt$$