We now proceede to complete the proof of:

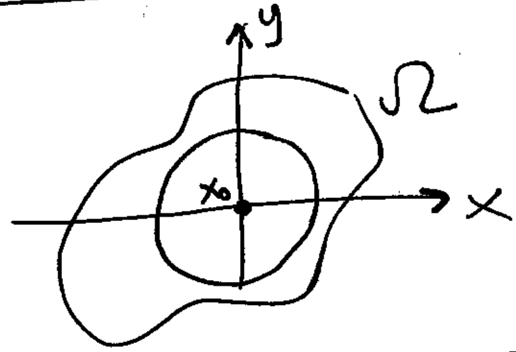
Theorem: If $f \in W^{1,2}_{loc}(\Omega)$ is weakly harmonic, then f is continuous in Ω and

$$f(x_0) = \fint_{B(x_0,r)} f(y)\, d\lambda(y)$$

whenever $\overline{B}(x_0,r) \subset \Omega$.

At this point, the only missing piece is the proof of:

$$\boxed{\text{Claim 1: } F(r) = \int_{\partial B(x_0,1)} f(r,z)\, d\mathcal{H}^{n-1}(z) \text{ is constant for all } 0 < r < d(x_0, \Omega)}$$



WLOG: $x_0 = 0$.

In order to prove Claim 1 we recall that, since $f \in W^{1,2}_{loc}(\Omega)$ is weakly harmonic, then:

(A) $\int_\Omega f \Delta \varphi \, d\lambda(x) = 0$, for all $\varphi \in C_c^\infty(\Omega)$.

We want to choose an appropiate test function φ in (A). We consider a test function of the form:

$$\varphi(x) = w(|x|)$$

A direct calculation shows:

$$\frac{\partial \varphi}{\partial x_i} = w'(|x|) \frac{\partial}{\partial x_i} (x_1^2 + \dots + x_n^2)^{1/2}$$

$$= w'(|x|) \frac{1}{2} (x_1^2 + \dots + x_n^2)^{-1/2} (2x_i)$$

$$= w'(|x|) \frac{x_i}{|x|}$$

and

$$\frac{\partial^2 \varphi}{\partial x_i^2} = \frac{x_i}{|x|} \left(w''(|x|) \frac{x_i}{|x|} \right) + w'(|x|) \left[\frac{|x| - x_i \cdot \frac{x_i}{|x|}}{|x|^2} \right]$$

$$= \frac{x_i^2}{|x|^2} w''(|x|) + w'(|x|) \left[\frac{|x|^2 - x_i^2}{|x|^3} \right]$$

$$\frac{\partial^2 \varphi}{\partial x_1^2} + \dots + \frac{\partial^2 \varphi}{\partial x_n^2} = w''(|x|) + \frac{n}{|x|} w'(|x|) - \frac{w'(|x|)}{|x|}$$

$$\therefore \quad \boxed{\Delta \varphi(x) = w''(|x|) + \frac{(n-1)}{|x|} w'(|x|)}$$

Let $r = |x|$.

Let
$0 < t < T < d(x_0, \partial\Omega)$

We choose $\omega(r)$ such that:
$$\omega \in C_c^\infty(t, T).$$

With such ω we can compute:

$$0 = \int_\Omega f(x)\,\Delta\varphi(x)\,d\lambda(x)$$

$$= \int_\Omega f(x)\left[\omega''(|x|) + \frac{(n-1)}{|x|}\omega'(|x|)\right]d\lambda(x)$$

$$= \int_t^T \int_{\partial B(0,1)} f(r,z)\left[\omega''(r) + \frac{(n-1)}{r}\omega'(r)\right] r^{n-1}\,d\mathcal{H}^{n-1}(z)\,dr$$

$$= \int_t^T \int_{\partial B(0,1)} f(r,z)\left[\omega''(r)\,r^{n-1} + (n-1)r^{n-2}\omega'(r)\right] d\mathcal{H}^{n-1}(z)\,dr$$

$$= \int_t^T \int_{\partial B(0,1)} f(r,z)\left(r^{n-1}\omega'(r)\right)'\,d\mathcal{H}^{n-1}(z)\,dr$$

$$= \int_t^T F(r)\,[r^{n-1}\omega'(r)]'\,dr$$

$$\therefore \boxed{0 = \int_t^T F(r)\,[r^{n-1}\omega'(r)]'\,dr} \quad (B)$$

How do we construct $w(r)$?

Let $\psi \in C_c^\infty(t,T)$ s.t.

$$\int_t^T \psi(r)\, dr = 0$$

For each real number r, define:

$$y(r) = \int_t^r \psi(s)\, ds$$

and define η by:

$$\eta(r) = \int_t^r \frac{y(s)}{s^{n-1}}\, ds$$

Finally, let:

$$w(r) = \eta(r) - \eta(T)$$

Note that $w \equiv 0$ on $[0,t]$ and $[T,\infty)$.

Indeed, if $T_0 > T$:

$$w(T_0) = \eta(T_0) = \int_t^{T_0} \frac{y(s)}{s^{n-1}}\, ds$$

$$= \int_t^T \frac{y(s)}{s^{n-1}}\, ds + \int_T^{T_0} \frac{y(s)}{s^{n-1}}\, ds$$

$$= \eta(T) + \int_{T_0}^{T} \frac{1}{s^{n-1}} \cdot 0\, ds$$

$= \eta(T)$; since $y(s) \equiv 0$, if $s > T$.

Since $w'(r) = \frac{y(r)}{s^{n-1}}$

we obtain from (B):

$$0 = \int_t^T F(r) \left[r^{n-1} \frac{y(r)}{r^{n-1}} \right]' dr$$

$$= \int_t^T F(r)\, y'(r)\, dr$$

$$= \int_t^T F(r)\, \psi(r)\, dr$$

We have:

$$\int_t^T F(r)\psi(r)\, dr = 0, \quad \text{for} \tag{C}$$

every $\psi \in C_c^\infty(t,T)$, $\int_t^T \psi = 0$

We consider the function $F(r)$ as a distribution, say T, defined on $C_c^\infty(t,T)$. The derivative of the distribution T is defined for any $\varphi \in C_c^\infty(t,T)$:

$$T'(\varphi) = -T(\varphi') = -\int_t^T F(r)\, \varphi'(r)\, dr$$

But notice that:

$$\int_t^T \varphi'(r) = \varphi(T) - \varphi(t) = 0.$$

Hence $\varphi' \in C_c^\infty(t,T)$, $\int_t^T \varphi' = 0$ and therefore from (C):

$$\int_t^T F(r)\,\varphi'(r)\,dr = 0$$

$$\boxed{\begin{array}{ll} \therefore & T'(\varphi) = 0 \quad \forall \varphi \in C_c^\infty(t,T). \\ \therefore & T' = 0 \end{array}}$$

We have shown in a previous lecture that if T a distribution then:

$T' = 0 \Rightarrow T$ is associated to a constant

Therefore, we have shown that:

$\exists \alpha$ s.t $F(r) = \alpha$, for all $t < r < T$

Since t and T are arbitrary, we conclude that $F(r) = \alpha$ for all $0 < r < d(x_0, \partial\Omega)$. ∎