We will finally complete the proof of:

Theorem 2: A weakly harmonic function $f \in W^{1,2}(\Omega)$ is of class $C^\infty(\Omega)$.

Recall that we have proved the following

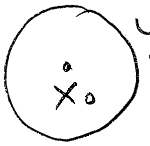
Theorem: If $f \in W^{1,2}_{loc}(\Omega)$ is weakly harmonic, then $f$ is continuous in $\Omega$ and

$$\boxed{f(x_0) = ⨍_{B(x_0,r)} f(y)\,dy ,} \qquad (A)$$

whenever $\overline{B}(x_0,r) \subset \Omega$

We need the following.

Definition: A function $\varphi : B(x_0,r) \to \mathbb{R}$ is called radial relative to $x_0$ if:
$\varphi$ is constant on $\partial B(x_0,t)$, $t \le r$.




$\varphi \equiv \alpha$ on $\partial B(x_0, t)$

We can prove:

<u>Corollary</u>: Suppose $f \in W^{1,2}(\Omega)$ is weakly harmonic and $\overline{B}(x_0, r) \subset \Omega$. If $\varphi \in C_c^\infty(B(x_0,r))$ is nonnegative and radial relative to $x_0$ with:

$$\int_{B(x_0,r)} \varphi(x)\,dx = 1$$

then

$$\boxed{f(x_0) = \int_{B(x_0,r)} f(x)\varphi(x)\,dx} \quad (B)$$

<u>Proof</u>: WLOG assume $x_0 = 0$.

Recall the following:

Lemma ✱: If $f$ is nonnegative and measurable, then:

$$\int_X f\,d\mu = \int_0^\infty \mu(\{x : f(x) > t\})\,d\lambda(t),$$

Note: We are considering here the measure space $(X, \mathcal{M}, \mu)$, with $\mu$ a positive measure.

Note, since $\varphi$ is radial then:

$$\boxed{\{x : \varphi(x) > t\} = B(0, r(t))} \quad (C)$$

Let:

$$M = \sup_{x \in B(0,r)} \varphi(x)$$

We have:

$$\int_{B(0,r)} f(x)\varphi(x)\,dx = \int_{B(0,r)} \left(f^{+}(x) - f^{-}(x)\right)\varphi(x)\,d\lambda(x)$$

$$= \int_{B(0,r)} \varphi(x) f^{+}(x)\,d\lambda(x) - \int_{B(0,r)} \varphi(x) f^{-}(x)\,d\lambda(x)$$

We define the measures (positive) as:

$$\mu_{+}(E) = \int_E f^{+}(x)\,d\lambda(x), \quad \mu_{-}(x) = \int_E f^{-}(x)\,d\lambda(x)$$

Hence, from Lemma $*$:

$$\int_{B(0,r)} f(x)\varphi(x)\,dx = \int_0^M \mu_{+}(\{x : \varphi(x) > t\})\,d\lambda(t) - \int_0^M \mu_{-}(\{x : \varphi(x) > t\})\,d\lambda(t)$$

Hence:

$$\int_{B(0,r)} f(x)\,\varphi(x)\,d\lambda(x) = \int_0^M \int_{\{\varphi>t\}} f^+(x)\,d\lambda(x)\,d\lambda(t)$$

$$- \int_0^M \int_{\{\varphi>t\}} f^-(x)\,d\lambda(x)\,d\lambda(t)$$

$$= \int_0^M \int_{\{\varphi>t\}} (f^+(x) - f^-(x))\,d\lambda(x)\,d\lambda(t)$$

$$= \int_0^M \int_{B(0,r(t))} f(x)\,d\lambda(x)\,d\lambda(t) \; ; \quad \text{by (C)}$$

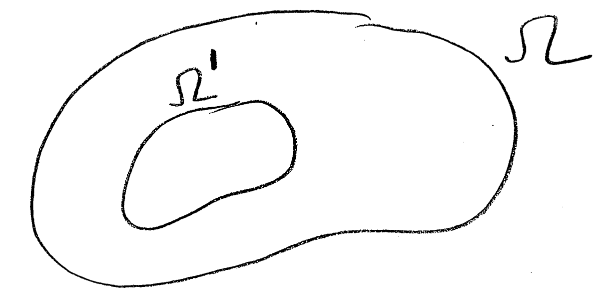
$$= \int_0^M f(0)\,\lambda[B(0,r(t))]\,d\lambda(t) \; ; \quad \text{by (A)}$$

$$= f(0) \int_0^M \lambda[\{\varphi>t\}]\,d\lambda(t)$$

$$= f(0) \int_{B(0,r)} \varphi(x)\,d\lambda(x) = f(0)\cdot 1 = f(0).$$

We have shown (B).

Proof of Theorem 2:




Let $\Omega' \subset\subset \Omega$.

Let $\varphi \in C_c^\infty(B(0,1))$ with:

$\int_{B(0,1)} \varphi(x)\, d\lambda(x) = 1$, $\varphi$ radial with respect to $0$.

Consider, for $x \in \Omega'$:

$$f * \varphi_\varepsilon(x) = \int_\Omega \varphi_\varepsilon(x-y) f(y)\, dy = \int_{B(x,\varepsilon)} \varphi_\varepsilon(x-y) f(y)\, dy$$

$$= \int_{B(0,\varepsilon)} \varphi_\varepsilon(y)\, f(x-y)\, dy$$

By Exercise 11.13, $h(y) := f(x-y)$ is weakly harmonic in $B(0,\varepsilon)$. Therefore, from (B) we have; since $\int_{B(0,\varepsilon)} \varphi_\varepsilon\, d\lambda(y) = 1$ and $\varphi_\varepsilon$ is radial;

$$f * \varphi_\varepsilon(x) = h(0) = f(x)$$