Def: The weak topology on a normed linear space X is the smallest topology on X with respect to which each $f \in X^*$ is continuous.

That such a weak topology exists may be seen by observing that the intersection of any family of topologies for X is a topology for X. In particular, the intersection of all topologies for X that contains all sets of the form

$$f^{-1}(U) \quad \text{where } f \in X^*, \ U \subset \mathbb{R} \text{ open}$$

is precisely the weak topology.

Notation: Let X be a normed linear space with norm $\|\cdot\| : X \to \mathbb{R}$. This norm induces a topology, called the strong topology that we will denote by (X, τ_s). We will denote the weak topology by (X, τ_w).

Note: A sequence $\{x_k\}$ in X converges weakly (i.e. with respect to the topology τ_w) to $x \in X$ if and only if

$$\lim_{k\to\infty} f(x_k) = f(x) \quad \forall f \in X^*$$

Notation $x_k \rightharpoonup x \iff f(x_k) \to f(x) \quad \forall f \in X^*$

Thm: X normed linear space
$x_k \rightharpoonup x$. Then:

(i) $\{\|x_k\|\}$ is bounded
(ii) $x \in \overline{W}$ (in the strong topology)
(iii) $\|x\| \leq \liminf_{k\to\infty} \|x_k\|$.

Proof:

(i) Let $f \in X^*$
$\{f(x_k)\}$ is convergent in $\mathbb{R} \Rightarrow$
$\sup\{|f(x_k)| : k=1,2,\dots\} < \infty$

$$\therefore \sup_{1\leq k<\infty} |\Phi(x_k)(f)| < \infty \quad \forall f \in X^*$$

X^* Banach space + Uniform Boundedness Principle (Thm 283.1)

$$\Rightarrow \sup_{1\leq k<\infty} \|\Phi(x_k)\| < \infty$$

Since

$$\|\Phi(x_K)\| = \|x_K\| \quad \forall K, \text{ then:}$$

$$\sup_{1 \le K < \infty} \{\|x_K\|\} < \infty.$$

Thm 283.1 (Uniform Boundedness Principle):
Let $\mathcal{F}$ be a family of continuous linear mapping from a Banach space X into a normed linear space Y s.t

$$\sup_{T \in \mathcal{F}} \|T(x)\| < \infty, \text{ for each } x \in X.$$

Then

$$\sup_{T \in \mathcal{F}} \|T\| < \infty.$$

(ii) Let $\overline{W} = \text{Closure}(W)$ in τ_s.
If $x \in \overline{W} \Rightarrow \exists f \in X^*$ s.t. $f(x) = 1$,
$f \equiv 0$ on $\overline{W}$.
$f(x_K) = 0 \ \forall K \Rightarrow f(x) = \lim_{K \to \infty} f(x_K) = 0$,
which contradicts $f(x) = 1$. Thus $x \in \overline{W}$

(iii) Note

$$|f(x)| = \lim_{K \to \infty} |f(x_K)| \le \liminf_{K \to \infty} \|f\| \|x_K\|, \ \forall f \in X^*$$

$$\therefore \ |f(x)| \le \liminf_{K \to \infty} \|x_K\| \ \forall f \in X^*, \|f\| = 1$$

$$\therefore \ \|x\| = \sup\{|f(x)| : f \in X^*, \|f\| = 1\} \le \liminf_{K \to \infty} \|x_K\|.$$

Thm: If X is a reflexive Banach space and Y is a closed subspace of X, then Y is a reflexive Banach space.

Thm If X is a reflexive Banach space, the the closed unit ball

$$B := \{ x \in X : \|x\| \leq 1 \}$$

is sequentially compact in the weak topology.

Sketch of Proof:

Assume first X is separable. Then:

$X \sim X^{**} \Rightarrow X^{**}$ separable $\Rightarrow X^{*}$ separable

Let $\{f_m\}_{m=1}^{\infty}$ be dense in X^{*}.

Let $\{x_k\} \subset B$. Thus $\|x_k\| \leq 1$

* $\{f_1(x_k)\}$ bounded in $\mathbb{R}$ $\Rightarrow \exists \{x_k^1\}$ subsequence of $\{x_k\}$ such that $\{f_1(x_k^1)\}$ converges in $\mathbb{R}$.

* $\{f_2(x_k^1)\}$ bounded $\Rightarrow \exists \{x_k^2\}$ subsequence of $\{x_k^1\}$ such that $\{f_2(x_k^2)\}$ converges.

* Continuining in this way. obtain

$$\begin{array}{llllll}
f_1(x_1^1) & f_1(x_2^1) & f_1(x_3^1) & f_1(x_4^1) & \cdots & \to \gamma_1 \\
f_2(x_1^2) & f_2(x_2^2) & f_2(x_3^2) & f_2(x_4^2) & \cdots & \to \gamma_2 \\
f_3(x_1^3) & f_3(x_2^3) & f_3(x_3^3) & f_3(x_4^3) & \cdots & \to \gamma_3 \\
\vdots & & & & & \\
f_m(x_1^m) & f_m(x_2^m) & f_m(x_3^m) & f_m(x_4^m) & \cdots & \to \gamma_m \\
\vdots & & & & &
\end{array}$$

Notice that:

$\{x_k^m\}$ is a subsequence of $\{x_k^{m-1}\}$

Set:

$$y_k = x_k^k$$

$\{y_k\}$ subsequence of $\{x_k\}$

(A) $\{f_m(y_k)\}$ converges as $k \to \infty$ $\forall m = 1, 2, \ldots$

Fix $f \in X^*$. $\{f_m\}$ dense in X^* $\Rightarrow$

$\exists m$ s.t. $\|f_m - f\| < \frac{\varepsilon}{4M}$

$M = \sup_{k \geq 1} \|x_k\| < \infty$:

Then:

$$|f(y_k) - f(y_\ell)| \le |f(y_k) - f_m(y_k)| + |f_m(y_k) - f_m(y_\ell)| + |f_m(y_\ell) - f(y_\ell)|$$

$$\le \|f - f_m\| \, \|y_k\| + \|f_m - f\| \, \|y_\ell\| + \underbrace{|f_m(y_k) - f_m(y_\ell)|}_{\text{Cauchy}}$$

$$\le \frac{\varepsilon}{4M} M + \frac{\varepsilon}{4M} \cdot M + \frac{\varepsilon}{2} \quad (\text{by (A)})$$

$$< \varepsilon$$

$\therefore$ $\{f(y_k)\}$ is Cauchy. Define:

$$\alpha(f) = \lim_{k \to \infty} f(y_k) \quad \forall f \in X^*$$

$\alpha : X \to \mathbb{R}$ is linear.

$$|\alpha(f)| \le \lim_{k \to \infty} |f(y_k)| \le \|f\| \, \|y_k\| \le M \|f\|.$$

$\therefore \quad \|\alpha(f)\| \le \overset{=1}{M} \|f\| \quad \forall f \in X^*$

$\therefore \quad \alpha$ is bounded

$\therefore \quad \alpha \in (X^*)^*$

$\Rightarrow \quad \exists x \in X \quad$ s.t. $\underline{\Phi(x) = \alpha}$.

$\therefore \quad \Phi(x)(f) = f(x) = \alpha(f) = \lim\limits_{k \to \infty} f(y_k),$

for all $f \in X^*$. $\|\alpha\| \le 1 \Rightarrow x \in B$

$$\boxed{\begin{array}{ll} \therefore & f(y_k) \to f(x) \quad \forall f \in X^* \\ \therefore & y_k \rightharpoonup x \quad (\text{in } \tau_w) \end{array}}$$

Assume now X is reflexive, not necessarily separable.

Let $\{x_k\} \subset B$. Let $W = \text{Span}\{x_k\}$

$Y = \overline{W}$ (in τ_s)

$$\left.\begin{array}{l} Y \text{ separable} \\ Y \text{ reflexive (Thm 291.1)} \end{array}\right\} \Rightarrow \begin{array}{c} \exists \{x_{k_j}\} \\ \text{s.t.} \\ x_{k_j} \rightharpoonup x \in Y \end{array}$$

$\therefore\ g(x_{k_j}) \to g(x) \quad \forall g \in Y^*$

Let $f \in X^*$, define:

$$f_Y = f|_Y \ , \quad f_Y \in Y^*$$

$$f(x) = f_Y(x) = \lim_{j\to\infty} f_Y(x_{k_j}) = \lim_{j\to\infty} f(x_k)$$

$\therefore\ f(x_k) \to f(x) \quad \forall f \in X^*$

$\therefore\ x_k \rightharpoonup x$

Thm 290.1 $\Rightarrow$

$$\|x\| \le \liminf \|x_k\| \le 1$$

$\therefore\ x \in B$.