(8.44)

Def:

The weak* topology on X^*, (X^*, τ_{w^*}), is defined as the smallest topology on X^* with respect to which each linear functional

$$w \in \Phi(X) \subset X^{**}$$

is continuous. We have:

$$f_K \xrightarrow{w^*} f \iff f_K(x) \to f(x) \quad \forall x \in X$$

Note: Since X^* is a normed linear space, we have these other two topologies in X^*:

$$(X^*, \tau_s), \quad (X^*, \tau_w)$$

where τ_s is the strong topology induced by $\|f\| = \sup\{f(x): x \in X, \|x\| = 1\}$, and τ_w is the weak topology in X^*, that is, the smallest topology such that each $f \in (X^*)^*$ is continuous.

Note: X reflexive $\Longrightarrow$ $\tau_{w^*} = \tau_w$
Topologies in X^*

Thm (Alaoglu's Theorem).

Suppose X is a normed linear space. The unit ball:

$$B := \{f \in X^* : \|f\| \le 1\}$$

of X^* is compact in the weak* topology.

Proof:

Set $I_x = [-\|x\|, \|x\|]$, $\forall x \in X$.

Tychonoff's Theorem 52.2 $\Rightarrow$

$$P = \prod_{x \in X} I_x$$

is compact with the product topology.

Note that:

$$B = \{f \in X^* : \|f\| \le 1\}$$
$$= \{f \in X^* : f(x) \in I_x \ \forall x \in X\}$$

Then

$$B \subset X^* \underset{\substack{\exists G \\ 1\text{-}1\text{- onto}}}{\longleftrightarrow} B' \subset P$$

Indeed, by definition, the Cartesian Product $P = \prod_{x \in X} I_x$ of the family of sets

$$\mathcal{P} = \{ I_x : x \in X \}$$

is the set of all mappings:

$$\beta : X \to \bigcup I_x$$

with the property that

$$\beta(x) \in I_x \quad \forall x \in X$$

Each mapping β is called the choice mapping of the family $\mathcal{P}$.

$\beta(x)$ is the x^{th} coordinate of β.

With this definition, it is clear that there is a 1-1 correspondence between B and a subset B' of B

$$G : B \subset X^* \longrightarrow B' \subset P$$

$$G(f) = \beta_f \qquad \forall f \in B.$$

with

$$\beta_f(x) = f(x)$$

$\beta \in B' \iff \beta(x) \in I_x \ \forall x$ and β linear.

8.47

The relative topology induced on B' by the product topology coincides with the relative topology induced on B by the weak* topology.

Therefore:

B' compact $\Longleftrightarrow$ B compact
(in the product topology) in the weak* topology

Claim: B' is compact in the product topology.

Let's prove first that B' is closed.

Let $\beta \in$ Closure (B'). Then $\forall \varepsilon > 0$, $\exists\ \alpha \in B'$ s.t:

$$|\beta(x) - \alpha(x)| < \varepsilon \quad \forall x \in X$$

$$\therefore\ |\beta(x)| - |\alpha(x)| \leq \varepsilon$$

$$\Rightarrow |\beta(x)| \leq \varepsilon + |\alpha(x)| \leq \varepsilon + \|x\|$$

Since $\alpha \in B' \Rightarrow \alpha(x) \in I_x$

8.48

$\therefore \quad |\beta(x)| \le \varepsilon + \|x\| \quad \forall \varepsilon, \forall x.$

$$\boxed{\begin{array}{l} \therefore \quad |\beta(x)| \le \|x\| \quad \forall x \in X \\ \therefore \quad \beta(x) \in I_x \quad \forall x \in X \end{array}} \qquad (*)$$

From $(*)$, to conclude $\beta \in B'$ we only need to check that β is linear.

Let $x, y \in X$, $c_1, c_2 \in \mathbb{R}$, $z = c_1 x + c_2 y$

Then $\forall \varepsilon > 0 \quad \exists \alpha \in B'$ s.t.

$$|\beta(x) - \alpha(x)| < \varepsilon, \quad |\beta(y) - \alpha(y)| < \varepsilon$$

$$|\beta(z) - \alpha(z)| < \varepsilon.$$

Since α linear $\Rightarrow$

$$\begin{aligned} &|\beta(z) - c_1\beta(x) - c_2\beta(y)| \\ &\le |\beta(z) - \alpha(z)| + |\alpha(z) - c_1\beta(x) - c_2\beta(y)| \\ &\le \varepsilon + |c_1\alpha(x) + c_2\alpha(y) - c_1\beta(x) - c_2\beta(y)| \\ &\le \varepsilon + |c_1|\,|\alpha(x) - \beta(x)| + c_2\,|\alpha(y) - \beta(y)| \\ &\le \varepsilon + \varepsilon|c_1| + \varepsilon|c_2| = \varepsilon(1 + |c_1| + |c_2|) \end{aligned}$$

ε arbitrary $\Rightarrow$

$$\beta(z) - c_1\beta(x) - c_2\beta(y) = 0$$

$\therefore \quad \boxed{\beta \text{ linear}} \Rightarrow \beta \in B'$

Since B' is closed and $B' \subset P$ with P compact, Proposition 39.1 yields that B' is compact. ∎

The previous theorem used:

<u>Prop</u> 39.1. Let (X, τ) be a topological space. If A and K are respectively closed and compact subsets of X with $A \subset K$, then A is compact.

<u>Thm 52.2</u>. (Tychonoff's Product Theorem). If $\{X_\alpha : \alpha \in A\}$ is a family of compact topological spaces and $X = \prod_{\alpha \in A} X_\alpha$ with the product topology, then X is compact.

Ex: Let $1 < p < \infty$.

Then Thm 29.2 implies that the unit ball

$$B = \{f \in L^p : \|f\|_p \leq 1\} \subset L^p(\mathbb{R}^n, \lambda)$$

is sequentially compact in the weak topology.

Thus, if $\{f_k\}$ is a sequence in $L^p(\mathbb{R}^n)$ such that:

$$\|f_k\|_p \leq M \quad \forall k = 1, 2, 3 \ldots$$

Then, there exists a subsequence $\{f_{k_j}\}$ of $\{f_k\}$ and a function $f \in B$ such that

$$f_{k_j} \rightharpoonup f. \qquad (*)$$

But $(*)$ is equivalent to:

$$F(f_{k_j}) \to F(f) \quad \forall F \in L^p(\mathbb{R}^n)^*$$

$$\therefore \int_{\mathbb{R}^n} g f_{k_j} \, d\lambda \to \int_{\mathbb{R}^n} g f \, d\lambda \quad \forall g \in L^{p'}(\mathbb{R}^n).$$

Ex: If we apply now Alaouglu's Theorem for $X = L^1(\mathbb{R}^n, \lambda)$; the since

$$(L^1(\mathbb{R}^n, \lambda))^* = L^\infty(\mathbb{R}, \lambda)$$

it follows that the unit ball

$$B = \{\Psi(f) \in (L^1(\mathbb{R}^n, \lambda))^* : \|f\|_\infty \leq 1\} \subset L^1(\mathbb{R}^n, \lambda)^*$$

is compact in the weak * topology of $L^1(\mathbb{R}^n, \lambda)^*$.

From this it follows that if $\{f_K\} \subset L^\infty(\mathbb{R}^n, \lambda)$ satisfies

$$\|f_K\|_\infty \leq M, \quad K = 1, 2, 3, \ldots$$

then there exists a subsequence $\{f_{K_j}\}$ of $\{f_K\}$ and $f \in L^\infty(\mathbb{R}^n, \lambda)$ such that

(*) $\Psi(f_{K_j}) \to \Psi(f)$ in the weak* topology of $L^1(\mathbb{R}^n)^*$

(*) is equivalent to

$$\Psi(f_{K_j})(g) \to \Psi(f)(g) \quad \forall g \in L^1(\mathbb{R}^n, \lambda)$$

$$\therefore \int_{\mathbb{R}^n} f_{K_j}\, g \, d\lambda \to \int_{\mathbb{R}^n} f g \, d\lambda \quad \forall g \in L^1(\mathbb{R}^n, \lambda)$$