Hilbert spaces.

An inner product on a linear space X is a function:

$$\langle \cdot , \cdot \rangle : X \times X \to \mathbb{R} \quad \text{s.t}$$

$$\langle x, y \rangle = \langle y, x \rangle$$
$$\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$$
$$\langle x, x \rangle \geq 0$$
$$\langle x, x \rangle = 0 \iff x = 0$$

<u>Thm</u> : $\|x\| := \sqrt{\langle x, x \rangle}$ defines a norm on X.

A Hilbert space is a linear space with an inner product that is <u>complete</u> with respect to the norm induced by the inner product.

Any Hilbert space is reflexive and therefore the set

$$\{ x \in H : \|x\| \leq 1 \}$$

is compact in the weak topology.

<u>Thm</u>: (Riesz Representation Theorem)
Let H be a Hilbert space.
Then, $\forall f \in H^*$, $\exists$ unique $y \in H$ s.t.:

$$f(x) = \langle y, x \rangle \quad \forall x \in H$$

With this correspondance H and H^* are isometrically isomorphic

<u>Proof</u>: Let $f \in H^*$

If $f = 0$ then choose $y = 0$.

Assume $f \neq 0$.

$\Rightarrow \quad M = \{x \in H : f(x) = 0\} \subset H$

(1) $M \neq H$
M closed

$\Rightarrow H = M \oplus M^\perp$, $M^\perp = \{x : \langle x, y \rangle = 0 \ \forall y \in M\}$

(1) $\Rightarrow \exists \ x_0 \in M^\perp$, $x_0 \neq 0$

$x_0 \notin M \Rightarrow f(x_0) \neq 0$.

8.64

Note:

$$f\left(x - \frac{f(x)}{f(x_0)} x_0\right) = 0 \quad , \forall x \in H$$

$$\therefore \quad x - \frac{f(x)}{f(x_0)} x_0 \in M \quad \forall x \in H$$

$$\therefore \quad \left\langle x - \frac{f(x)}{f(x_0)} x_0, x_0 \right\rangle = 0 \quad \forall x$$

$$\langle x, x_0 \rangle - \frac{f(x)}{f(x_0)} \langle x_0, x_0 \rangle = 0$$

$$\langle x, x_0 \rangle - \|x_0\|^2 \frac{f(x)}{f(x_0)} = 0$$

$$\therefore \quad f(x) = \frac{f(x_0)}{\|x_0\|^2} \langle x, x_0 \rangle \quad \forall x \in H$$

Set $\boxed{y = \frac{f(x_0)}{\|x_0\|^2} x_0}$

With this choice of y it is clear that

$$\boxed{f(x) = \langle x, y \rangle \quad \forall x \in H}$$

8.65

We claim:

$$\|y\| = \|f\|$$

$$|f(x)| = |\langle x, y\rangle|$$

$$\leq \|x\| \, \|y\| \text{ , by Schwarz's inequality}$$

$$\therefore \|f\| \leq \|y\|$$

But $\left|f\left(\frac{y}{\|y\|}\right)\right| = \frac{1}{\|y\|} |f(y)|$

$$= \frac{1}{\|y\|} |\langle y, y\rangle|$$

$$= \frac{\|y\|^2}{\|y\|} = \|y\|$$

$$\therefore \|f\| = \|y\|$$

y is unique :

Assume

$$\langle y_1, x\rangle = \langle y_2, x\rangle \quad \forall x \in H$$

$$\therefore \langle y_1 - y_2, x\rangle = 0 \quad \forall x \in H$$

$$\therefore \langle y_1 - y_2, y_1 - y_2\rangle = 0$$

$$\therefore \|y_1 - y_2\|^2 = 0$$

$$\therefore y_1 = y_2.$$

Many theorems (versions) are named: "Riesz Representation Theorem".

RRT1: H Hilbert space. Then, for each $f \in H^*$, $\exists$ unique $y \in H$ such that

$$f(x) = \langle y, x \rangle \quad \forall x \in H.$$

Under this correspondance H and H^* are isometrically isomorphic. That is, the map:

$$\Psi : H \to H^*$$

$$\Psi(y)(x) = \langle y, x \rangle \quad \forall x \in H$$

is one-to-one, linear, on-to, and

$$\|\Psi(y)\| = \|y\|, \quad \forall y \in H.$$

(8.67)

(2) RRT2: Let $1 \le p < \infty$, $(X, \mathcal{M}, \mu)$ a measure space such that μ is σ-finite.

For every $F \in (L^p(X))^*$, $\exists$ unique $g \in L^{p'}(X)$, such that

$$F(f) = \int_X fg\, d\mu \quad \forall f \in L^p(X)$$

Under this correspondance $L^{p'}(X)$ and $(L^p(X))^*$ are isometrically isomorphic.

That is, the map

$$\Upsilon : L^{p'}(X) \longrightarrow (L^p(X))^*$$

$$\Upsilon(g)(f) = \int_X fg\, d\mu \quad \forall f \in L^p(X)$$

is one-to-one, linear, on-to and

$$\|\Upsilon(g)\| = \|g\|_{p'} \quad \forall g \in L^{p'}(X)$$