Lecture 12

Sets of finite perimeter

If $E$ is a set with $C^1$-boundary, the following Gauss–Green theorem is proved in Chapter 9 of the textbook:

**Theorem 1:** (Gauss–Green theorem) If $E$ is an open set with $C^1$-boundary, then:

$$
\int_E \nabla \psi(x) \, dx = \int_{\partial E} \psi \nu_E \, d\mathcal{H}^{n-1}, \quad \forall \psi \in C_c^1(\mathbb{R}^n)
$$

where $\nu_E$ is the exterior unit normal to $E$.

Equivalently, the divergence theorem holds true:

$$
\int_E \text{div} \mathbf{T}(x) \, dx = \int_{\partial E} \mathbf{T} \cdot \nu_E \, d\mathcal{H}^{n-1}, \quad \forall \mathbf{T} \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)
$$

This theorem is the starting motivation to study sets of finite perimeter. Indeed, we say that a Lebesgue measurable set $E \subseteq \mathbb{R}^n$ is a set of locally finite perimeter if $E \in \mathcal{P}$, a Radon measure on $\mathbb{R}^n$ such that:

$$
\int_E \nabla \psi(x) \, dx = \int_{\mathbb{R}^n} \psi \, d\mu_E, \quad \forall \psi \in C_c^1(\mathbb{R}^n)
$$
The total variation measure $|\mu_E|$ of $\mu_E$ induces the notion of relative perimeter $P(E; F)$ of $E$ with respect to a set $F \subset \mathbb{R}^n$, and of total perimeter $P(E)$ of $E$, as:

$$P(E; F) = |\mu_E|(F) \quad P(E) = |\mu_E|(\mathbb{R}^n),$$

In particular, $E$ is a set of finite perimeter if and only if $P(E) < \infty$.

For example, if $E$ is an open set with $C^1$ boundary with outer unit normal $n_E \in C(\partial E, S^{n-1})$ then from the Gauss-Green formula in Theorem 4 it follows that $E$ is a set of locally finite perimeter with:

$$\mu_E = n_E \mathcal{H}^{n-1}\partial E, \quad |\mu_E| = \mathcal{H}^{n-1}\partial E,$$

$$P(E; F) = \mathcal{H}^{n-1}(F \cap \partial E), \quad P(E) = \mathcal{H}^{n-1}(\partial E).$$

Therefore, in the next six or seven lectures we will show that these definitions lead to a geometrically meaningful generalization of the notion of open set with $C^1$-boundary, with natural and powerful applications to the study of geometric variational problems.
Since, given $\mathbb{E} \subset \mathbb{R}^n$ Lebesgue measurable, we want to produce a measure $\mu_E$ such that the Gauss-Green theorem holds, then it is natural to think of the Riesz representation theorem as the main analytical tool to produce $\mu_E$. Thus, with the Riesz theorem in mind, we can now define:

**Definition:** Let $\mathbb{E} \subset \mathbb{R}^n$ Lebesgue measurable. We say that $\mathbb{E}$ is a set of locally finite perimeter in $\mathbb{R}^n$ if for every compact set $K \subset \mathbb{R}^n$ we have:

$$\sup \left\{ \int_{\mathbb{E}} \text{div} T(x) \, dx : T \in C_c^1(\mathbb{R}^n; \mathbb{R}^n), \text{spt} T \cap K, \sup_{\mathbb{R}^n} |T| \leq 1, \right\} < \infty$$

If this quantity is bounded independently of $K$, then we say that $\mathbb{E}$ is a set of finite perimeter in $\mathbb{R}^n$.

With this definition and using the Riesz representation theorem, we can prove the Gauss-Green formula for $\mathbb{E}$.

**Theorem 2:** Let $\mathbb{E} \subset \mathbb{R}^n$ Lebesgue measurable. Then $\mathbb{E}$ is a set of locally finite perimeter $\iff \exists \mu_E$, a $\mathbb{R}^n$-valued Radon measure on $\mathbb{R}^n$ such that:

$$\int_{\mathbb{E}} \text{div} T = \int_{\mathbb{R}^n} T \cdot d\mu_E \quad \forall T \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$$

Moreover, $\mathbb{E}$ is a set of finite perimeter $\iff |\mu_E(\mathbb{R}^n)| < \infty$.
Remark 1: Note that:
\[ \int_E \text{div} \mathbf{T} = \int_{\mathbb{R}^n} \mathbf{T} \cdot d\mu_E \]
is equivalent to
\[ \int_E \nabla \psi = \int_{\mathbb{R}^n} \psi d\mu_E, \quad \psi \in C_c^1(\mathbb{R}^n) \]
Indeed:
Let \( \psi \in C_c^1(\mathbb{R}^n) \). Let \( \mathbf{T}_i = (0, \ldots, \psi_i, \ldots, 0) \) in position.
\[ \Rightarrow \int_E \text{div} \mathbf{T}_i = \int_{\mathbb{R}^n} \mathbf{T}_i \cdot d\mu_E \]
\[ \therefore \int_E \psi x_i = \int_{\mathbb{R}^n} \psi (d\mu_E)_i, \quad i = 1, 2, \ldots, n \]
since \( \nabla \psi = (\psi x_1, \ldots, \psi x_n) \Rightarrow \)
\[ \int_E \nabla \psi = \int_{\mathbb{R}^n} \psi d\mu_E \]
on the other hand:
Let \( \mathbf{T} \in C_c^1(\mathbb{R}^n, \mathbb{R}^n) \), \( \mathbf{T} = (\psi_1, \ldots, \psi_n) \). Since:
\[ \int_E \nabla \mathbf{T}_i = \int_{\mathbb{R}^n} \mathbf{T}_i \cdot d\mu_E \Rightarrow \int_E (\psi_i)_i = \int_{\mathbb{R}^n} \psi_i (d\mu_E)_i, \]
for \( i = 1, 2, \ldots, n \). Hence:
\[ \sum_{i=1}^{n} \int_E (\psi_i)_i x_i = \sum_{i=1}^{n} \int_{\mathbb{R}^n} \psi_i (d\mu_E)_i \]
\[ \therefore \int_E \text{div} \mathbf{T} = \int_{\mathbb{R}^n} \mathbf{T} \cdot d\mu_E \]
**Definition:** We call $\mu_E$ the Gauss-Green measure of $E$, and define the relative perimeter of $E$ in $FCIR^n$, and the perimeter of $E$, as:

$$P(E; F) = \mathcal{W}_E(F), \quad P(E) = |\mu_E|(\mathbb{R}^n).$$

**Proof of Theorem 2:**

Let $E$ be a set of locally finite perimeter. Define:

$$L: C_c^1(\mathbb{R}^n; \mathbb{R}^m) \to \mathbb{R}$$

$$\langle L, T \rangle = \int_E \text{div}_E T(x) \, dx.$$

Let $K \subset \mathbb{R}^n$ compact,

$$\sup \{ |\langle L, T \rangle| : T \in C_c^1(\mathbb{R}^n; \mathbb{R}^m), \text{sp}(T) \subset K, |T| \leq 1 \} < \infty$$

because, by definition of set of finite perimeter:

$$|\langle L, T \rangle| = \left| \int_E \text{div}_E T(x) \, dx \right| \leq C(K), \quad \forall T \in C_c^1(\mathbb{R}^n; \mathbb{R}^m)$$

$$|T| \leq 1.$$

Hence, $L$ is continuous in $C_c^1(\mathbb{R}^n; \mathbb{R}^m)$, with respect to the topology in $C_c(\mathbb{R}^n; \mathbb{R}^n)$ introduced in Lecture 4. Hence, $L$ can be extended by density to a bounded continuous linear functional on $C_c(\mathbb{R}^n; \mathbb{R}^n)$. By Riesz's theorem $\exists \mu_E$ such that:

$$\langle L, T \rangle = \int_{\mathbb{R}^n} T : d\mu_E \Rightarrow \int_E \text{div}_E T(x) \, dx = \int_{\mathbb{R}^n} T : d\mu_E$$

The converse implication is trivial. $\blacksquare$
Remark 2: Let $E$ be a set of locally finite perimeter in $\mathbb{R}^n$. If $|E \Delta F| = 0$, then:

$$\int_E \text{div} T = \int_F \text{div} T, \quad \int_E \text{div} T = \int_{\mathbb{R}^n} T \cdot d\mu_E \quad \forall T \in C_0^1(\mathbb{R}^n; \mathbb{R}^n)$$

Thus, $\int_{\mathbb{R}^n} T \cdot d\mu_E \quad \forall T \in C_0^1(\mathbb{R}^n; \mathbb{R}^n) \Rightarrow F$ is of locally finite perimeter.

Hence, there exists $F$ such that

$$\int_{\mathbb{R}^n} T \cdot d\mu_E \quad \forall T \in C_0^1(\mathbb{R}^n; \mathbb{R}^n)$$

In view of Remark 1, and since $C_0^1(\mathbb{R}^n)$ is dense in $C_c(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} T \cdot d\mu_E \quad \forall T \in C_c(\mathbb{R}^n) \Rightarrow \mu_E = \mu_F$$

(See Lecture 4, page 4.5).

We can actually modify $E$ in such a way that the new set $F$ has a "huge" topological boundary but still $\mu_F = \mu_E$. For example, let $E \subset \mathbb{R}^2$ be the unit disk and $F = E \cup Q^2$. Thus, $|E \Delta F| = 0$, but $\mu_F = \mu_E + 2\mu_{\mathbb{R}^2 \setminus E}$. Or $F$ can be as follows:

$$E = \text{disk} \ E \ \text{minus all the curves in the picture} \ \ \ |E \Delta F| = 0$$

Actually, by the Gauss-Green theorem, note that if $E \subset \mathbb{R}^n$ is open (not necessarily bounded) with $C^1$ boundary, then $E$ is a set of locally finite perimeter with $\mu_E = \mu_{E \Delta \mathbb{R}^n \setminus \mathbb{R}^n}$, $\mathbb{R}_E = \mathbb{R}^{n-1}(\mathbb{R}^n \setminus \mathbb{R}^n)$, and $P(E, F) = \chi^{n-1}(\partial E \setminus \partial F) \forall F \subset \mathbb{R}^n$. 
In chapter 9 of the textbook, the Gauss-Green formula is proved to be true also for sets \( E \) with Lipschitz boundary or polyhedral boundary. Hence such sets \( E \) are of locally finite perimeter, with \( P(E; F) = \mathcal{H}^{n-1}(F \cap \partial E) \) whenever \( E \subset \mathbb{R}^n \).

Moreover, if \( E \) is bounded, then \( E \) is of finite perimeter.

Since convex sets have locally Lipschitz boundary, it follows that convex sets are of locally finite perimeter, while bounded convex sets are of finite perimeter.

**Remark 3**: Recall from the theory of distributions that if \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \), then \( f \) induces a distribution \( T_f \) defined as \( \langle T_f, \psi \rangle = \int_{\mathbb{R}^n} f \psi \, dx \).

Moreover, the derivative of the distribution \( T_f \) is:

\[
\langle DT_f, \psi \rangle = -\int_{\mathbb{R}^n} f \nabla \psi \, dx, \quad \forall \psi \in C_0^\infty(\mathbb{R}^n).
\]

Then, if \( E \subset \mathbb{R}^n \) is Lebesgue measurable \( \Rightarrow X_E \in L^1_{\text{loc}}(\mathbb{R}^n) \)

Hence:

\( E \subset \mathbb{R}^n \) is a set of locally finite perimeter \( \iff \) the distributional gradient \( DX_E \) can be represented as the integration with respect to \( -\mu_E \).
Lower semicontinuity of perimeter

We say that $E_i$ locally converges to $E$ ($E_i \to E$) if

$$\chi_{E_i} \to \chi_E \quad \text{in} \quad L^1_{\text{loc}}(\mathbb{R}^n)$$

That is,

$$\lim_{i \to \infty} \int \chi_{K \cap (E_i \Delta E)} \, dx = 0, \quad \forall K \subset \mathbb{R}^n \text{ compact}$$

We simply say that $E_i$ converges to $E$, $E_i \to E$, if $\chi_{E_i} \to \chi_E$ in $L^1(\mathbb{R}^n)$; that is:

$$\lim_{i \to \infty} \int |E \Delta E_i| = 0$$

Remark 4: Let $E$ be a set of locally finite perimeter. Then, Theorem 2 implies $\mathcal{M} \subsetneq \mathcal{M}_E$ Radon s.t.

$$\int_E \text{div} T(x) \, dx = \int_{\mathbb{R}^n} T \cdot d\mathcal{M}_E \quad \forall T \in C_c^1(\mathbb{R}^n; \mathbb{R}^n),$$

and by our study of the Riesz's theorem in Lecture 4 (See page 4.3) we have; for $A$ open:

$$\mathcal{P}(E, A) = |M_E|(A) = \sup \{ \int_A \text{div} T(x) \, dx : T \in C_c^\infty(A; \mathbb{R}^n), \|T\|_L^\infty \leq 1 \}$$

Note: $C_c^\infty, C_c^1$ are both dense in $C_c$
Theorem 3 (Lower semicontinuity of perimeter):

Let \( \{E_i\} \) sequence of sets of locally finite perimeter in \( \mathbb{R}^n \), with

\[
E_i \xrightarrow{loc} E ; \quad \limsup_{i \to \infty} P(E_i ; K) < \infty \quad \forall K \subset \mathbb{R}^n \text{ compact.}
\]

Then:

(a) \( E \) is a set of locally finite perimeter in \( \mathbb{R}^n \)
(b) \( ME_i \xrightarrow{*} ME \)
(c) \( P(E; A) \leq \liminf_{i \to \infty} P(E_i; A) \), \( \forall A \subset \mathbb{R}^n \) open.

Proof:

By Remark 4; for \( T \in \mathcal{C}_c^0 (A; \mathbb{R}^n) \), \( |T| \leq 1 \), \( A \) open:

\[
\int_E \nabla T(x) \cdot dx = \lim_{i \to \infty} \int_{E_i} \nabla T(x) \cdot dx = \lim_{i \to \infty} \int_{\mathbb{R}^n} T \cdot dME_i \leq \liminf_{i \to \infty} |ME_i| (A)
\]

\[\therefore \text{ } E \text{ is a set of locally finite perimeter (using } A=K \text{ and hypothesis)}\]

\[\therefore \text{ Per } (E; A) \leq \liminf_{i \to \infty} P(E_i; A) \; \text{ even for } A \text{ unbounded.}\]

Now, since \( E_i \xrightarrow{loc} E \), we have:

\[
\int_{E_i} \nabla \varphi \cdot dx \to \int_{E} \nabla \varphi \cdot dx \;
\]

\[
: \int_{\mathbb{R}^n} \varphi \cdot dE_i \to \int_{\mathbb{R}^n} \varphi \cdot dE, \quad \forall \varphi \in \mathcal{C}_c^0 (\mathbb{R}^n)
\]

Since \( \mathcal{C}_c (\mathbb{R}^n) \) is dense in \( \mathcal{C}_c (\mathbb{R}^n) \) we have

\[
\int_{\mathbb{R}^n} \varphi \cdot dE_i \to \int_{\mathbb{R}^n} \varphi \cdot dE, \quad \forall \varphi \in \mathcal{C}_c (\mathbb{R}^n) ; \text{ i.e. } ME_i \xrightarrow{*} ME.
\]
As explained in Remark 2, we can modify a set of locally finite perimeter $E$ by a set of $L^n$-measure zero without changing its Gauss–Green measure, and, as a consequence, its perimeter. Such modifications may largely increase the topological boundary. The following lemma shows how to modify $E$ to “minimize” the size of the topological boundary.

Lemma 1: If $E$ is a set of locally finite perimeter in $\mathbb{R}^n$, then:

\[
\text{spt } M_E = \{ x \in \mathbb{R}^n : 0 < |E \cap B(x,r)| < w_n r^n, \forall r > 0 \} \cap E
\]

Moreover, there exists a Borel set $F$ such that:

\[ |E \Delta F| = 0, \quad \text{spt } M_F = \partial F \]

Proof: If $x \in \mathbb{R}^n$, $|E \Delta B(x,r)| = 0$, for some $r > 0$, then

\[
\int_E \Psi \, d\nu = \int_{\mathbb{R}^n} \Psi \, d\nu_E, \quad \forall \Psi \in C^\infty_c (B(x,r))
\]

\[
\int_{E \cap B(x,r)} \Psi \, d\nu = 0
\]

\[
\therefore \int_{\mathbb{R}^n} \Psi \, d\nu_E = 0 \quad \forall \Psi \in C^\infty_c (B(x,r)) \quad \Rightarrow \quad |M_E| (B(x,r)) = 0 \quad \Rightarrow \quad x \notin \text{spt } M_E
\]
If \( x \in \mathbb{R}^n \) and \( 1E \cap B(x,r) \cap E \) for some \( r > 0 \), then \( j \) for \( \psi \in C_0^\infty (B(x,r)) \):

\[
\int_E \nabla \psi = \int_{\mathbb{R}^n} \psi \, dm_E
\]

\[
\int_{E \cap B(x,r)} \nabla \psi
\]

\[
\int_{B(x,r)} \nabla \psi = 0
\]

\[
\int_{\mathbb{R}^n} \psi \, dm_E = 0, \quad \forall \psi \in C_0^\infty (B(x,r)) \Rightarrow |E| \leq (B(x,r)) = 0
\]

\[
\Rightarrow x \notin \text{Spt} \, m_E
\]

Also, if \( x \notin \text{Spt} \, m_E \Rightarrow |E| \leq (B(x,r)) = 0 \), some \( r > 0 \), and, for \( \psi \in C_0^\infty (B(x,r)) \):

\[
0 = \int_{\mathbb{R}^n} \psi \, dm_E = \int_E \nabla \psi = \int_{\mathbb{R}^n} \chi_E \nabla \psi.
\]

By Lemma 7.5 in textbook \( (u \in L_1^1 \text{ loc} (\mathbb{R}^n), A \) open connected, \( \int u \nabla \psi = 0, \forall \psi \in C_c^\infty (A) \Rightarrow u = c \in \mathbb{R} \) a.e. in \( A \) ) it follows that:

\[
\chi_E = c \quad \text{a.e. on } B(x,r)
\]

\[
\Rightarrow |E \cap B(x,r)| \leq |E|, \quad \omega_n r^n.
\]

\[
\Rightarrow \text{Spt} \, m_E = \{ x \in \mathbb{R}^n : 0 < |E \cap B(x,r)| < \omega_n r^n \quad \forall r > 0 \cap E \}
\]
To find $F$, WLOG $E$ is Borel (by regularity of $\mathbb{L}^n$). Define:

$A_0 := \{ x \in \mathbb{R}^n : \exists r > 0 \text{ s.t. } |E \cap B(x, r)| = 0 \}$

$A_1 := \{ x \in \mathbb{R}^n : \exists r > 0 \text{ s.t. } |E \cap B(x, r)| = \omega n r^n \}$

Let $\{ x_i \} \subset A_0$, $A_0 \subset \bigcup_{i=1}^{\infty} B(x_i, r_i)$, $r_i > 0$, $|E \cap B(x_i, r_i)| = 0$.

$\Rightarrow |E \cap A_0| = 0$

$\Rightarrow |A_1 \cap E| = 0$; since $\mu_{\mathbb{R}^n \setminus E} = -\mu_E$.

$A_1$ for $E$ is $A_0$ for $\mathbb{R}^n \setminus E$.

Exercise 12.09 in textbook.

Define Borel set:

$F := (A_1 \cup E) \setminus A_0$

With:

$|F \setminus E| \leq |A_1 \setminus E| = 0$, $|E \setminus F| \leq |E \cap A_0| = 0$

$\therefore |E \Delta F| = 0$.

By $(\star)$:

$spt \mu_F = spt \mu_E = \mathbb{R}^n \setminus (A_0 \cup A_1) \subset \partial F$

On the other hand, $\partial F \subset spt \mu_F$ because:

$A_1 \subset \partial F$ (by construction), $F \subset \mathbb{R}^n \setminus A_0$.

We conclude:

$spt \mu_F = \partial F$. 

\[ \]