Lecture 17

In this lecture we will prove Steiner inequality (Theorem 2, Lecture 16), which was the main ingredient in the proof of the isoperimetric inequality.

Step one: In this step we show that

\[ P(E^5) \leq P(E) \]

Since \( P(E) \leq \infty \), \( |E| \leq \infty \), by Corollary 2 in Lecture 16, there exists \( \{E_k\}_{k=1}^\infty \) bounded open sets with polyhedral boundary such that, as \( k \to \infty \),

\[ E_k \to E \quad \text{in} \quad L^1(\mathbb{R}^n), \quad P(E_k) \to P(E) \quad (1) \]

Let:

\[ m_k(z) = \mathcal{L}^1((E_k)_z), \quad m(z) = \mathcal{L}^1(E_z) \]

\[ G_k = \{ z \in \mathbb{R}^{n-1} : m_k(z) > 0 \} \]

\[ D_k = \{ z \in \mathbb{R}^{n-1} : (E_k)_z \text{ is not an interval} \} \]

\[ G = \{ z \in \mathbb{R}^{n-1} : m(z) > 0 \} \]

Note that \( \gamma_{E_k} \) takes only finitely many values, and hence, up to rotating each \( E_k \) by a rotation sufficiently close to the identity, we can assume

\[ \gamma_{E_k} \cdot e_n \neq 0 \quad (2) \]
We will prove below that a bounded set with polyhedral boundary satisfying (2) has the following properties:

\[
P(E_K^S) \leq P(E_K)
\]

\[
2\alpha n^{-1}(D_K)^2 \leq P(E_K)(P(E) - P(E_K^S))
\]

(3)

By Fubini's Theorem:

\[
|E_K \Delta E| = \int_{\mathbb{R}^{n-1}} \left| (E_K)_z \Delta E_z \right| dz = \int_{\mathbb{R}^{n-1}} \left| m_K(z) - m(z) \right| dz = \int_{\mathbb{R}^{n-1}} \left| E_K^S \Delta E^S \right|
\]

Thus, \( |E_K \Delta E| \to 0 \) yields \( E_K^S \to E^S \) in \( L^1(\mathbb{R}^n) \)

\[
\Rightarrow P(E^S) \leq \liminf_{K \to \infty} P(E_K^S)
\]

(4)

Since \( P(E_K) \to P(E) \); from (3) and (4):

\[
2 \limsup_{K \to \infty} \alpha n^{-1}(D_K)^2 \leq P(E) \left( \limsup_{K \to \infty} P(E_K) + \limsup_{K \to \infty} (P(E_K^S)) \right)
\]

\[
= P(E) \left( P(E) - \liminf_{K \to \infty} P(E_K^S) \right)
\]

\[
\leq P(E) \left( P(E) - P(E^S) \right)
\]

Thus:

\[
2 \limsup_{K \to \infty} \alpha n^{-1}(D_K)^2 \leq P(E) \left( P(E) - P(E^S) \right)
\]

(5)
Clearly, from (5):

\[ P(E^s) \leq P(E) \]

which is (\text{****}) in Theorem 2.

We now prove (i) in Theorem 2. Indeed, if \( P(E) = P(E^s) \) then:

\[ \limsup_{k \to \infty} \chi_{n-1}(D_k) \leq 0; \]

that is,

\[ \lim_{k \to \infty} \chi_{n-1}(D_k) = 0 \implies \chi_{D_k} \to 0 \text{ in } L'(\mathbb{R}^{n-1}) \quad (6) \]

Now:

\[ \int_{\mathbb{R}^{n-1}} f'(E_k) \Delta E_z \, dz = 0 \implies 0 \text{ as } k \to \infty \text{ implies that there exists a subsequence of } \{ E_k \}, \text{ denoted again as } \{ E_k \}, \text{ such that:} \]

\[ f'(E_k) \Delta E_z \to 0 \text{ for a.e. } z \in \mathbb{R}^{n-1}. \]

\[ \therefore \chi_{(E_k)z} \to \chi_{E_z} \text{ in } L'(\mathbb{R}), \text{ for a.e. } z \in \mathbb{R}^{n-1} \quad (7) \]

And also:

\[ \chi_{G_k} \to \chi_{G} \text{ in } L'(\mathbb{R}^{n-1}) \quad (8) \]
Now:
\[ \chi_{(E_k)_z} \to \chi_{E_z} \quad \text{in } L^1(\mathbb{R}) \quad \text{implies} \]
\[ P(E_z) \leq \lim \inf_{k \to \infty} P((E_k)_z), \quad \text{and this is} \]
\[ \quad \text{true for a.e. } z. \]

From (6) and (8):
\[ \chi_{G_k \setminus D_k} \to \chi_G \quad \text{in } L^1(\mathbb{R}^{n-1}) \]
\[ \text{recall that} \]
\[ \chi^{n-1}(D_k) \to 0 \quad \text{and} \]
\[ \text{thus} \quad \chi^{n-1}
\begin{align*}
&= \chi^{n-1}(G_k \cap (\mathbb{R}^n \setminus D_k)) \\
&\downarrow \\
&\chi^{n-1}(G) \quad \text{as } k \to \infty
\end{align*}

Note that \( \chi^{n-1}(D_k) \to 0 \) means that as \( k \to \infty \), "most" of the sections \( (E_k)_z \) are intervals.

From \( \chi_{G_k \setminus D_k} \to \chi_G \) in \( L^1(\mathbb{R}^{n-1}) \), we have that, for a further subsequence:
\[ \lim_{k \to \infty} \chi_{G_k \setminus D_k}(z) = \chi_G(z), \quad \text{a.e. } z. \]

Thus; multiplying by \( \chi_G(z) \) in above inequality:
\[ \chi_G(z) P(E_z) \leq \chi_G(z) \lim \inf_{k \to \infty} P((E_k)_z) \]
\[ = \left( \lim_{k \to \infty} \chi_{G_k \setminus D_k}(z) \right) \left( \lim \inf_{k \to \infty} P((E_k)_z) \right), \quad \text{a.e. } z \]
\[ \leq \lim \inf_{k \to \infty} \left( \chi_{G_k \setminus D_k}(z) P((E_k)_z) \right), \quad \text{a.e. } z \]
where we have used:
\[
\liminf_{k \to \infty} (a_k b_k) \geq \liminf_{k \to \infty} a_k \liminf_{k \to \infty} b_k.
\]

We have:
\[
\chi_{G_k}(z) P(E_k) \leq \liminf_{k \to \infty} \chi_{G_k \setminus D_k}(z) P((E_k)_z), \text{ a.e. } z
\]
\[
\Rightarrow \int_G P(E_k) \, dz \leq \int_{G_k \setminus D_k} \liminf_{k \to \infty} P((E_k)_z) \, dz
\]
\[
\leq \liminf_{k \to \infty} \int_{G_k \setminus D_k} P((E_k)_z) \, dz ; \text{ Using Fatou's lemma}
\]
\[
= \liminf_{k \to \infty} \int_{G_k \setminus D_k} 2 \, dz ; \text{ Since } (E_k)_z \text{ is an interval}
\]
\[
= 2 \liminf_{k \to \infty} \mathcal{H}^{n-1}(G_k \setminus D_k) ; \text{ hence it has perimeter 2 in } \mathbb{R}.
\]
\[
= 2 \mathcal{H}^{n-1}(G).
\]
\[
\therefore \int_G P(E_k) \, dz \leq 2 \mathcal{H}^{n-1}(G) \quad (Q)
\]

We are going to use now the following proposition (see textbook):

**Proposition (Sets of finite perimeter in \( \mathbb{R} \)):** \( E \subset \mathbb{R} \)
is of locally finite perimeter if and only if it is equivalent to a countable union of (possibly unbounded) open intervals lying at mutually positive distance.
Clearly, if \( E \subseteq \mathbb{R}, \mathcal{L}^n(E) < \infty \) then 
\[ P(E) \geq 2 \] 

Thus, going back to (9):
\[ P(E_z) - 2 \geq 0, \text{ and hence:} \]
\[ \int_G (P(E_z) - 2) \, d\mathbb{H}^{n-1} = 0 \]

implies
\[ P(E_z) = 2 \text{ for a.e. } z \in G. \]

By proposition 1 we have that \( E_z \) is equivalent to a countable union of open intervals. But, since \( P(E_z) = 2 \), we conclude that such union consists of only one interval. Hence:
\( E_z \) is equivalent to an open interval, a.e. \( z \).

We have proved (i) in Theorem 2, but we are left to prove that (3) holds for any bounded set with polyhedral boundary:

We assume that \( \nu_E(x) \cdot e_n \neq 0, \forall x \in \partial E, \nu_E(x) \) exterior unit normal
We have:

\[ G = \bigcup_{i=1}^{M} G_i \]

and affine functions \( v_i^k, u_i^k : G_i \to \mathbb{R} \), \( 1 \leq i \leq M \), \( 1 \leq k \leq N(i) \), with

\[ \partial E = \bigcup_{i=1}^{M} \bigcup_{k=1}^{N(i)} \Gamma(v_i^k, G_i) \cup \Gamma(u_i^k, G_i), \]

\[ E = \bigcup_{i=1}^{M} \left\{ (z, t) \in G_i \times \mathbb{R} : t \in \bigcup_{k=1}^{N(i)} \left( v_i^k(z), u_i^k(z) \right) \right\} \]

Note:

- \( m(z) = \sum_{k=1}^{N(i)} u_i^k(z) - v_i^k(z) \), \( \forall z \in G_i \).
- \( m \) is continuous, piecewise affine.

Note: This partition exists because of the assumption \( v_E(x) \cdot e_n \neq 0 \), \( \forall x \in \mathbb{E} \) and the implicit function theorem.

We will use the following theorem (see Chapter 9 in textbook):

**Thm** (Area of a graph of codimension one). If \( u : \mathbb{R}^{n-1} \to \mathbb{R} \) is a Lipschitz function, then for every Lebesgue measurable set \( G \) in \( \mathbb{R}^{n-1} \),

\[ \mathcal{H}^{n-1}(\Gamma(u; G)) = \int_G \frac{\sqrt{1 + |\nabla u(z)|^2}}{dG} \]

(To prove this theorem, apply the area formula to the Lipschitz function \( f(z) = (z, u(z)) \), \( z \in \mathbb{R}^{n-1} \) and compute...
the Jacobian of \( f \) as
\[
Jf = \sqrt{(\nabla f)^T (\nabla f)}, \text{ which is } Jf = \sqrt{1 + (\nabla f)^T (\nabla f)}.
\]

Note that:
\[
E^s = \{(z,t) \in G \times \mathbb{R}, \ |t| < \frac{m(z)}{2}\}
\]
\( E^s \) is a bounded open set with polyhedral boundary.

Using the above formula to compute the area of a graph we have:
\[
P(E^s) = \mathcal{H}^{n-1}(\partial E^s) = 2 \int_G \sqrt{1 + |\nabla m|^2} = \sum_{i=1}^{M} \sqrt{A + 1m_i^2}.
\]
\[
P(E) = \sum_{i=1}^{M} \int_{G_i} \sum_{K=1}^{N(i)} \sqrt{1 + |\nabla u_z^K|^2 + \sqrt{1 + |\nabla v_c^K|^2}} \, dz
\]

Since \( z \mapsto \sqrt{1 + |z|^2} \) is convex we have:
\[
\sum_{K=1}^{N(i)} \sqrt{1 + |\nabla u_z^K|^2 + \sqrt{1 + |\nabla v_c^K|^2}} \geq 2 \sum_{K=1}^{N(i)} \sqrt{1 + \frac{|\nabla u_z^K - \nabla u_z^K|^2}{2}}
\]

Recall, \( f \) convex \( \Rightarrow f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y), \ \lambda = \frac{1}{2} \Rightarrow f(\frac{x+y}{2}) \leq \frac{1}{2}f(x) + \frac{1}{2}f(y) \)

\[
= 2 \sum_{i=1}^{N(i)} \left\{ \frac{1}{N(i)} \sum_{K=1}^{N(i)} \sqrt{1 + \frac{|\nabla u_z^K - \nabla u_z^K|^2}{2}} \right\}
\]

\( f \) convex, \( \lambda_1 + \lambda_2 + \cdots + \lambda p = 1 \) \( \Rightarrow f(\lambda_1 x_1 + \cdots + \lambda_p x_p) \leq \lambda_1 f(x_1) + \cdots + \lambda_p f(x_p) \)
\[
\geq 2 \sqrt{ \sum_{i=1}^{N(i)} \frac{1}{N(i)} \left( \sum_{K=1}^{N(i)} \frac{|\nabla u_z^K - \nabla u_z^K|^2}{2} \right) = \sqrt{4N(i)^2 + 1m_i^2}}
\]
Therefore:
\[
P(E) \geq \sum_{i=1}^{M} \int_{G_i} \sqrt{4N(i)^2 + |\nabla m|^2} \, dz
\]
and
\[
P(E^c) = \sum_{i=1}^{M} \int_{G_i} \sqrt{4 + |\nabla m|^2} \, dz
\]
Thus; since \( N(i) \geq 1 \):
\[
P(E^c) \leq P(E)
\]
Recall our notation:
\[
D = \{ z \in G : E_z \text{ is not an interval} \}
\]
\[
\therefore \quad N(i) \geq 2 \iff G_i \cap D \neq \emptyset
\]
Then:
\[
P(E) - P(E^c) \geq \sum_{i=1}^{M} \int_{G_i \cap D} \sqrt{4N(i)^2 + |\nabla m|^2} - \sqrt{4 + |\nabla m|^2} \, dz
\]
\[
= \sum_{i=1}^{M} \int_{G_i \cap D} \frac{4(N(i)^2 - 1)}{\sqrt{4N(i)^2 + |\nabla m|^2} + \sqrt{4 + |\nabla m|^2}} \, dz
\]
\[
\geq 2 \sum_{i=1}^{M} \int_{G_i \cap D} \frac{1}{\sqrt{4N(i)^2 + |\nabla m|^2}} ; \quad \text{since } N(i) \geq 2
\]
By Holder inequality:
\[2 \mathcal{H}^{n-1}(D)^2 = 2 \left( \int_D \frac{(4N(i)^2 + 1\mathcal{V}m)^{1/4}}{(4N(i)^2 + 1\mathcal{V}m)^{1/4}} \right)^2 \]

\[
\leq 2 \left( \int_D \left( \sum_{i=1}^{M} \frac{1}{G_i \cap D \sqrt{4N(i)^2 + 1\mathcal{V}m}} \right)^{1/2} \right)^2 \left( \int_D \left( \sum_{i=1}^{M} \sqrt{4N(i)^2 + 1\mathcal{V}m} \right)^{1/2} \right)^2

\leq (P(E) - P(E^c))(P(E))

Thus, we have proved, for \( E \) a bounded set with polyhedral boundary and \( v_E \cdot e_n \neq 0 \) on \( \partial E \) that:

\[
2 \mathcal{H}^{n-1}(D)^2 \leq P(E)(P(E) - P(E^c))
\]

\[P(E^c) \leq P(E)\]

which justifies (3) in the proof of Theorem 2. In conclusion, we have shown that if \( E \subset \mathbb{R}^n \) is of finite perimeter, \( |E| < \infty \) then \( E^s \) satisfies \( P(E^s) \leq P(E) \) and, if \( P(E^s) = P(E) \), then \( E \subset \) is equivalent to an interval, for a.e. \( z \). The rest of the proof of Theorem 2 can be found in textbook.
This is another isoperimetric inequality that is not sharp.

**Proposition (A perimeter bound on volume):** If \( E \) is a bounded set of finite perimeter in \( \mathbb{R}^n \), \( n \geq 2 \), then
\[
P(E) \geq |E| \frac{n-1}{n}
\]

**Proof:** Following as in the proof of the Sobolev Embedding Theorem (see "Modern Real Analysis", chapter 11) we have:
\[
\|u\|_{L^{n/(n-1)}(\mathbb{R}^n)} \leq \|\nabla u\|_{L^1(\mathbb{R}^n; \mathbb{R}^n)} \quad \forall u \in C_\infty^0(\mathbb{R}^n)
\]

We now define:
\[
u_\varepsilon = \chi_{E_\varepsilon} \ast f_\varepsilon.
\]

Recall that:
\[
\int_{\mathbb{R}^n} |\nabla u_\varepsilon| \rightarrow P(E) \quad \text{as } \varepsilon \rightarrow 0
\]

Therefore:
\[
P(E) \frac{n}{n-1} = \lim_{\varepsilon \rightarrow 0} \|\nabla u_\varepsilon\|_{L^1(\mathbb{R}^n; \mathbb{R}^n)}^{n/(n-1)} \geq \liminf_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n} |u_\varepsilon|^{n/(n-1)} \quad \text{by above}
\]
\[
\geq \int_{\mathbb{R}^n} \liminf_{\varepsilon \rightarrow 0} |u_\varepsilon|^{n/(n-1)} \quad \text{by Sobolev inequality}
\]
\[
\geq \int_{\mathbb{R}^n} \lim_{\varepsilon \rightarrow 0} |u_\varepsilon|^{n/(n-1)} \quad \text{by Fatou's Lemma}
\]
\[
= \int_{\mathbb{R}^n} \chi_E = |E|.
\]
We look at the following application of isoperimetric inequalities:

**Cheeger Sets**: Let $p > 0$, $n > 2$

A open set in $\mathbb{R}^n$.

The $p$-cheeger problem in $A$ is the variational problem:

$$c(p, A) = \inf \left\{ \frac{P(E)}{|E|_p} : E \subset A \right\} \quad (**)$$

A minimizer $E$ of $(**)$ is called a $p$-cheeger set of $A$.

- If $p < \frac{n-1}{n}$, by scaling $c(p, A) = 0$ and hence $p$-cheeger sets can not exist.

- If $p > \frac{n-1}{n}$ and $A$ is bounded, then $p$-Cheeger sets exist (Use the Direct method and the isoperimetric inequality $|E|_p \frac{n-1}{n} \leq P(E)$).

- If $p = \frac{n-1}{n}$, then by the Isoperimetric inequality (Theorem 1 in Lecture 16) it follows that balls contained in $A$ are the (only) $p$-Cheeger sets in $A$. 