Lecture 19

Continuation of proof of Theorem 1 (Tangential properties of the reduced boundary). See Lecture 18.

Step two: We prove two lower bounds on the $n$-density ratios of $E$ and $\mathbb{R}^n \setminus E$ at $x \in \mathbb{R}^n$:

$$\frac{|E \cap B(x, r)|}{r^n} \geq \frac{1}{(3n)^n} \quad \forall r < r(x) \quad (1)$$

$$\frac{|(\mathbb{R}^n \setminus E) \cap B(x, r)|}{r^n} \geq \frac{1}{(3n)^n} \quad \forall r < r(x). \quad (2)$$

Recall that $E$ and $\mathbb{R}^n \setminus E$ satisfy:

$$\mu_E = -\mu_{\mathbb{R}^n \setminus E},$$

and hence

$$\mathcal{E}^*E = \mathcal{E}^*(\mathbb{R}^n \setminus E).$$

Thus, we only need to prove (1). Define:\n
$$m(r) = |E \cap B(x, r)|, \quad r > 0$$

By coarea formula:

$$m(r) = |E \cap B(x, r)| = \int_0^r \mathcal{H}^{n-1}(E \cap B(x, t)) \, dt. \quad (3)$$

From (3) we have:

$m$ is absolutely continuous and:

$$m'(r) = \mathcal{H}^{n-1}(E \cap B(x, r)) \text{ for a.e. } r > 0.$$
Note that $m(r) > 0$, $r > 0$, $m(0) = 0$. Indeed if $m(r) = 0$ for some $r > 0$ where the following holds:

$$P(E \cap B(x, r)) = \chi_{n-1}(E \cap \overline{B}(x, r))$$

$$+ P(E; B(x, r)),$$

then, since $m(r) = 0 \Rightarrow P(E \cap B(x, r)) = 0$. Then, for the equality to hold we need:

$$P(E; B(x, r)) = 0,$$

but $x \notin E \Rightarrow x \notin \text{supp } m_E \Rightarrow m_E(B(x, r)) = 0 \Rightarrow P(E; B(x, r)) > 0$. We conclude that $m(r) > 0$ for a.e. $r$, but since $m$ is continuous and increasing, $m(r) > 0$, $\forall r > 0$. Then

$$m(r)^{\frac{n-1}{n}} = |E \cap B(x, r)|^{\frac{n-1}{n}}$$

$$\leq P(E \cap B(x, r)); \text{ See Lecture 17, Page 17.11}$$

$$\leq 3 \chi_{n-1}(E \cap \overline{B}(x, r)); \text{ by Step 1, for a.e. } r < r(x)$$

$$= 3 m'(r)$$

$$m(r)^{\frac{n-1}{n}} m'(r) \geq \frac{1}{3}$$

$$m(r)^{-1 + \frac{1}{n}} m'(r) \geq \frac{1}{3}$$

$$\frac{d}{dr} \left( n m(r)^{\frac{1}{n}} \right) \geq \frac{1}{3} \text{ a.e. } r < r(x)$$
Integrating both sides:
\[ n \, m(r)^{1/n} \geq \frac{1}{3} r , \quad \forall r < r(x) \]

\[ \Rightarrow m(r)^{1/n} \geq \frac{r}{3n} \]

\[ \Rightarrow m(r) \geq \frac{r^n}{(3n)^n} \Rightarrow |E \cap B(x,r)| \geq \frac{1}{(3n)^n} , r < r(x) \]

---

**Step three**:

To prove:
\[ \text{Ex}_r \xrightarrow{\text{loc}} H_x , \text{ as } r \to 0^+ \]

it is enough to show that for every \( \{r_i\} \), \( r_i \to 0 \), \( \exists r_i \in \mathbb{R}^+ \) such that
\[ \text{Ex}_{r_i} \xrightarrow{\text{loc}} H_x , \text{ as } k \to \infty \]

**Note**: \[
P(\text{Ex}_r \cap B_R) = \frac{P(E \cap B(x,rR))}{r^{n-1}} \quad \text{we proved in Lecture 18 that} \]
\[ |\mu_{\text{Ex}_r}| = \frac{1}{r^{n-1}} |\phi_{x,r}| \frac{\# E}{\mu E} \]

\[ \leq \frac{C(n)(rR)^{n-1}}{r^{n-1}} , \quad \forall R < r(x) \text{ by step one} \]
\[
\leq \frac{C(n)R^{n-1}}{r} , \quad \forall r < r(x) \]

By compactness, (Lecture 14, Page 14.1),
\( \exists F \) set of locally finite perimeter, and a sub-
sequence \( \{E_{x, r_i^j}\} \) such that

\[
E_{x, r_i^j} \xrightarrow{j \to \infty} F, \quad M_{E_{x, r_i^j}} \xrightarrow{j \to \infty} M_F
\]

For simplicity, we denote \( \{E_{x, r_i^j}\}_{i=1}^\infty \) again as \( \{E_{x, r_i^j}\} \). Now, up to extracting a further subsequence:

\( \exists \lambda \) such that \( M_{E_{x, r_i^j}} \overset{*}{\rightharpoonup} \lambda \). Then

\[
\lim_{i \to \infty} M_{E_{x, r_i^j}}(BR) \to M_F(BR) \quad \text{a.e. } R > 0
\]

where \( \lambda(\partial BR) = 0 \)

Also, since \( x \in \mathcal{E}^+ \) and \( M_{E_{x, r}} = \frac{1}{m-1}(\Phi_{x, r})_+ \cdot M_E \):

\[
\lim_{r \to 0^+} \frac{M_{E_{x, r}}(BR)}{|M_{E_{x, r}}|(BR)} = \lim_{r \to 0^+} \frac{M_E(B(x, rR))}{|M_E|(B(x, rR))}
\]

\[= \nu_E(x)\]

\[
\therefore \lim_{r \to 0^+} \frac{P(E_{x, r}; BR)}{\nu_E(x) \cdot M_{E_{x, r}}(BR)} = 1
\]

Taking dot product with \( \nu_E(x) \) on both sides.

We have:

\[
P(F; BR) \leq \liminf_{i \to \infty} P(E_{x, r_i^j}; BR) \quad \text{lower semicontinuity}
\]

\[= \lim_{i \to \infty} \nu_E(x) \cdot M_{E_{x, r_i^j}}(BR)\]
\[ = \nu_E(x) \cdot m_F(B_R); \quad \text{by (4)} \]
\[ \leq |m_F(B_R)| \]
\[ \leq |m_F|(B_R) = P(F, B_R) \]

We have shown; for a.e. \( R \):
\[ |m_F|(B_R) = \lim_{i \to \infty} |m_{E_{x_i}B_i^c}|(B_R) \]
\[ \text{(5)} \]

By (5) and (4) we have:
\[ |m_{E_{x_i}B_i^c}| \overset{*}{\to} |m_F| \; ; \quad \text{(6)} \]

(6) is true by the fact that (See Exercise 4.31):
\[ m_k \overset{*}{\to} m \; \text{and} \; m_k(B_{r_i,j}) \to m(B_{r_i,j}) \; \forall j, \; r_i \to \infty = m_k \overset{*}{\to} m \]

By (5):
\[ 0 = \int_{B_R} (1 - \nu_E(x) \cdot \nu_F(y)) \, \text{d} |m_F|(y), \quad \text{for a.e. } R > 0 \]
\[ \Rightarrow \quad 1 - \nu_E(x) \cdot \nu_F(y) = 0 \quad \text{for } |m_F|-\text{a.e. } y \in \mathcal{E}^*F \]
\[ \Rightarrow \quad \nu_E(x) = \nu_F(y) \quad \text{for } |m_F|-\text{a.e. } y \in \mathcal{E}^*F \quad \text{(7)} \]
By Lemma 3, Lesson 18 we have that $F$ is equivalent to a half space. That is, for $\alpha \in \mathbb{R}$ such that

$$\int F \Delta \{ y \in \mathbb{R}^n : y \leq \alpha \} = 0.$$ 

We have two possibilities:

1. $\alpha < 0$. In this case $| \mathbb{F} \cap B_\alpha | = 0$, so that

$$0 = \frac{| \mathbb{F} \cap B_\alpha |}{|B_\alpha|} = \lim_{i \to \infty} \frac{|E_i \cap B_\alpha|}{|B_\alpha|} = \lim_{i \to \infty} \frac{|E_i \cap B(x, r_i \alpha)|}{|B(x, r_i \alpha)|},$$

which is a contradiction to Step 2.

2. $\alpha > 0$. The same argument gives a contradiction.

We conclude $\alpha = 0$ and hence $F = H_x$.

---

Step four: We have proved that as $r \to 0$:

$$\mu_{E \cap B_r} \to \mu_{H_x}, \quad E_{x_r} \xrightarrow{\text{loc}} H_x, \quad |M_{E \cap B_r}| \to |M_{H_x}|$$

But by the Gauss–Green Theorem:

$$\mu_{H_x} = \nu_{E(x)} \chi_{H_x^c} - L \mathcal{H}_x,$$

which concludes the proof of Theorem 1.
Let us now make a resume of important things we have learned so far:

- **M C IR^n, H^kLM Radon measure. M is locally H^k-rectifiable if ∃ \{ f_i \}; f_i : IR^k \to IR^n Lipschitz, such that:

  \[
  H^k(M \setminus \bigcup_{i=1}^\infty f_i'(IR^k)) = 0
  \]

- **Criterion for rectifiability:** M Borel set, \( \mu \) Radon measure on IR^n, \( \mu(\{IR^n \setminus M\}) = 0 \). If \( \forall x \in M \), \( \exists \Pi_x \) a k-dimensional hyperplane such that

  \[
  \frac{1}{r^k}(\Phi_{x,r})(\# \mu) \overset{*}{\rightharpoonup} H^kL \Pi_x,
  \]

  then M is locally H^k-rectifiable and:

  \[
  \mu = H^kLM.
  \]

Hypothesis can be written as (recall \( \Phi_{x,r}(y) = \frac{y-x}{r} \)):

\[
\frac{\mu(B(x,r))}{r^k} \to w_k \quad \text{as } \ r \to 0
\]

or \( \frac{\mu(B(x,r))}{w_k r^k} \to 1 \). Indeed, the weak convergence implies (since \( H^kL \Pi_x(B(0,1)) = 0 \)):

\[
\frac{1}{r^k}(\Phi_{x,r})(\# \mu)(B(0,1)) \to H^kL \Pi_x(B(0,1)),
\]

And \( (\Phi_{x,r})(\# \mu)(B(0,1)) = \mu(\Phi_{x,r}^{-1}(B(0,1))) = \mu(B(x,r)) \).
Sets of finite perimeter:

\[
\mu_{E,x,r} = \frac{1}{r^{n-1}} \int_{\mathbb{R}^n} \phi_{x,r} \, d\mu_E,
\]

\[
\text{Lemma 1, Lecture 18.}
\]

Note: In the proof of Lemma 1 we have shown that:

\[
\int_{E,x,r} \phi \, d\mu_E = \frac{1}{r^{n-1}} \int_{\mathbb{R}^n} \phi_{x,r} \, d\mu_E,
\]

But \( \phi \in C^1(\mathbb{B}(0,R)) \Rightarrow \phi_{x,r} \in C^1(\mathbb{B}(x,R)) \). Thus, taking the sup over all such \( \phi \):

\[
\| \mu_{E,x,r} \|_{\mathbb{B}(0,R)} = \frac{1}{r^{n-1}} \| \mu_E \|_{\mathbb{B}(0,R)};
\]

which is the same as:

\[
\lim_{r \to 0} \int_{\mathbb{B}(0,1)} \phi_{x,r} \, d\mu_E = \int_{\mathbb{B}(0,1)} \phi_{x,r} \, d\mu_E
\]

Theorem 1 says (Lecture 18, Page 18.5) if \( x \notin \partial E^* \) then:

(a) \( E_{x,r} \xrightarrow{loc} H_x \)

(b) \( \mu_{E,x,r} \xrightarrow{**} \nu_{E}(x) \cdot \Pi_x = H^{n-1} \cdot \Pi_x \)

We proved (Page 18.6) that:

(a) implies \( \frac{|E \cap \mathbb{B}(x,r)|}{|\mathbb{B}(x,r)|} \to \frac{1}{2} \Rightarrow \exists E \subset E^{'1/2} \)

(b) implies \( |\mu_{E,x,r}(\mathbb{B}(0,1))| = \frac{1}{r^{n-1}} |\mu_E|_{\mathbb{B}(x,r)} = \frac{P(E;\mathbb{B}(x,r))}{r^{n-1}} \to w_{n-1} \)

or \( \frac{P(E;\mathbb{B}(x,r))}{w_{n-1}} \to 1 \)

which is the same as:

\[
\lim_{r \to 0} \int_{\mathbb{B}(0,1)} \phi_{x,r} \, d\mu_E = \int_{\mathbb{B}(0,1)} \phi_{x,r} \, d\mu_E
\]
Since
\[
\lim_{r \to 0} \frac{M_E(B(x,r))}{w_{n-1}r^{n-1}} \to 1 \quad \forall x \in \partial^*E,
\]
the criterion for rectifiability implies that \(\partial^*E\) is locally \(H^{n-1}\)-rectifiable and \(|M_E| = H^{n-1} L \partial^*E\).

Since we now that \(M_E = \nu_E |M_E| L \partial^*E\), we obtain:
\[
M_E = \nu_E H^{n-1} L \partial^*E
\]

We have proved the following Corollary of Theorem 1:

Corollary III: If \(E\) is a set of (locally) finite perimeter, then \(\partial^*E\) is (locally) \(H^{n-1}\)-rectifiable and
\[
M_E = \nu_E H^{n-1} L \partial^*E.
\]

Moreover, the approximate tangent space to \(\partial^*E\) at \(x \in \partial^*E\) agrees with the orthogonal space to the measure-theoretic outer unit normal to \(E\) at \(x\), that is,
\[
T_x (\partial^*E) = \nu_E(x)^\perp.
\]
Federer's Theorem

We already know that:
\[ \partial^* E \subseteq E^{(1/2)} \]

**Definition:** The essential boundary \( \partial^e E \) of a Lebesgue measurable set \( E \subseteq \mathbb{R}^n \) is defined as:
\[ \partial^e E = \mathbb{R}^n \setminus (E^{(0)} \cup E^{(1/2)}) \].

Obviously:
\[ E^{(1/2)} \subseteq \partial^e E \].

We have

**Theorem (Federer's theorem):** If \( E \subseteq \mathbb{R}^n \) is a set of locally finite perimeter, then \( \partial^* E \subseteq E^{(1/2)} \subseteq \partial^e E \), with
\[ \mathcal{H}^{n-1}(\partial^e E \setminus \partial^* E) = 0 \]

**Proof:** The relative isoperimetric inequality says (we will prove later, see chapter 12 from textbook):
\[ P(E; B(x,r)) \geq c(n) \min \{ |E \cap B(x,r)|, |B(x,r) \setminus E| \}^{\frac{n-1}{n}} \]

\[ |E \cap B(x,r)| \leq w_n r^n \implies |E \cap B(x,r)|^{\frac{n-1}{n}} \leq w_n^{\frac{n-1}{n}} r^{n-1} \]

\[ \implies |E \cap B(x,r)|^{\frac{1}{n}} \leq w_n^{\frac{1}{n}} r \]

\[ \implies |E \cap B(x,r)|^{\frac{1}{n}} |E \cap B(x,r)|^{\frac{n-1}{n}} \leq w_n^{\frac{n-1}{n}} r \]

\[ \implies w_n^{\frac{1}{n}} r \leq |E \cap B(x,r)| \]

\[ \therefore w_n^{\frac{1}{n}} r \leq |E \cap B(x,r)| \]

\[ \therefore |E \cap B(x,r)|^{\frac{n-1}{n}} \geq |E \cap B(x,r)| \]
Therefore,
\[
(\text{A}) \quad \frac{P(E; B(x,r))}{r^{n-1}} \geq c(n) \min \left\{ \frac{|E \cap B(x,r)|}{r^n}, \frac{|B(x,r) \setminus E|}{r^n} \right\}
\]

If \( \theta_{n-1}^* (x^{-1} \mathcal{L}^* E, x) = \limsup_{r \to 0} \frac{\chi^{n-1} \left( \partial^* E \cap B(x,r) \right)}{w_{n-1} r^{n-1}} = 0 \),

then, by (A) above, \( x \in E^{(0)} \cup E^{(1)} \). Thus,
\( x \in \partial^* E \Rightarrow \theta_{n-1}^* (x^{-1} \mathcal{L}^* E, x) > 0 \),

\( \partial^* E \in \{ x \in \mathbb{R}^n : \theta_{n-1}^* (x^{-1} \mathcal{L}^* E, x) > 0 \} \) (B)

(Recall that \( x \in \partial^* E \Rightarrow \theta_{n-1}^* (x^{-1} \mathcal{L}^* E, x) = 1 \)).

From (B):
\( \partial^* E \cap \partial^* E \in \mathcal{F}_2 = \{ x \in \mathbb{R}^n \setminus \partial^* E : \theta_{n-1}^* (x^{-1} \mathcal{L}^* E) (x) > 0 \} \).

We now use a Corollary we proved in Lecture 7 (Page 7.4) and notice that we have already used this Corollary many times!

(Corollary: \( M \subset \mathbb{R}^n \), \( \chi^s (M \cap K) < \infty \) for any compact \( K \). Then
\[
\lim_{r \to 0} \frac{\chi^s (M \cap B(x,r))}{w_{s,r}^s} = 0 \text{ for } \chi^s \text{-a.e. } x \in \mathbb{R}^n \setminus M
\]

With \( M = \partial^* E \) and \( s = n-1 \) we get:
\[
\chi^{n-1} (F) = 0
\]

\( \therefore \chi^{n-1} (\partial^* E \setminus \partial^* E) = 0 \).