Lecture 2

Problem: \( \mathcal{M}(\mu) \) could be "too small" to work with, which leads us to introduce:

**Def:** \( \mathcal{B}(\mathbb{R}^n) = \text{Borel sets} = \text{smallest \( \sigma \)-algebra containing the open sets} \)

**Def:** \( \mu \) is a Borel measure if \( \mu \) is an outer measure \( \mu \) on \( \mathbb{R}^n \) such that \( \mathcal{B}(\mathbb{R}^n) \subseteq \mathcal{M}(\mu) \).

**Carathéodory criterion:** If \( \mu \) is an outer measure on \( \mathbb{R}^n \), then \( \mu \) is Borel if and only if
\[
\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2),
\]
for every \( E_1, E_2 \subset \mathbb{R}^n \), \( \text{dist}(E_1, E_2) > 0 \).

**Proof:** It suffices to show that:

\( \forall \) \( C \) closed, \( \mu(A) \geq \mu(A \cap C) + \mu(A \setminus C), \forall A \).

- \( \mu(A) = \infty \)
- Assume \( \mu(A) < \infty \), \( C_k := \{ x : d(x, C) \leq \frac{1}{k} \} \)
- \( R_k = (C_k \setminus C_{k+1}) \cap A \)

\[ d(C \setminus A, A \setminus C_k) > 0 \implies \mu(A \cap C) + \mu(A \setminus C_k) = \mu(A \setminus C_k \cup (A \setminus C_k)) \leq \mu(A) \]
\[ \mu(A \cap C) + \mu(A \setminus C) \leq \mu(A) + \sum_{j=k+1}^{\infty} \mu(R_j) \]

\[ \leq \mu(A) + \sum_{j=k+1}^{\infty} \mu(R_j) \]

\[ \sum_{j=1}^{N} \mu(R_j) = \sum_{j=1}^{N} \mu(R_{2j}) + \sum_{j=1}^{N} \mu(R_{2j-1}) \quad \text{dist}(R_{2j}, R_{2k}) > 0 \]

\[ = \mu(\bigcup_{j=1}^{N} R_{2j}) + \mu(\bigcup_{j=1}^{N} R_{2j-1}) \]

\[ \leq 2\mu(A) < \infty. \]

So \[ \sum_{j=1}^{N} \mu(R_j) < \infty \] and hence \[ \lim_{k \to \infty} \sum_{j=k}^{\infty} \mu(R_j) = 0. \]

**Ex:** \( \mathcal{H}^S \) is Borel, \( 0 \leq S \leq n \) is Borel on \( \mathbb{R}^n \).

Show first \( \mathcal{H}^S(E_1 \cup E_2) = \mathcal{H}^S(E_1) + \mathcal{H}^S(E_2) \) if \( \text{dist}(E_1, E_2) > 0 \). Then, let \( S \to 0 \).

**Ex:** \( L^n \) is Borel on \( \mathbb{R}^n \) (Recall that \( L^n = \mathcal{H}^n \)).

**Def:** \( \mu \) is a Borel regular measure if for every \( F \subset \mathbb{R}^n \) there exists a Borel set \( E \) such that:

\[ F \subset E, \quad \mu(E) = \mu(F). \]

**Thm:** \( \mathcal{H}^S \) is Borel regular on \( \mathbb{R}^n \). (\( L^n \) is also Borel regular with similar proof).

**Proof:** Use closed sets in the definition of \( \mathcal{H}^S \).
\[ \forall K \in \mathbb{N}, \exists \{F_i^K\}_{i=1}^{\infty} \text{ such that} \]
\[
\text{covering of } E \text{ such that:}
\]
\[
\text{diam} (F_i^K) \leq \frac{1}{K}, \quad \sum_{i=1}^{\infty} w_5 \left( \frac{\text{diam} F_i^K}{2} \right)^s \leq \mathcal{H}^s_{\chi K} (E) + \frac{1}{K}
\]
\[
E \subset \bigcup_{i=1}^{\infty} F_i^K.
\]

Let \( F = \bigcap_{k=1}^{\infty} \bigcup_{i=1}^{\infty} F_i^K \supseteq E \). Clearly, \( \mathcal{H}^s (E) \leq \mathcal{H}^s (F) \)

We only need \( \mathcal{H}^s (F) \leq \mathcal{H}^s (E) \):

\[
\mathcal{H}^s_{\chi K} (F) \leq \sum_{i=1}^{\infty} w_5 \left( \frac{\text{diam} F_i^K}{2} \right)^s \leq \mathcal{H}^s_{\chi K} (E) + \frac{1}{K}
\]

↑ def. of \( \mathcal{H}^s \)

Let \( K \to \infty \), \( \Rightarrow \mathcal{H}^s (F) \leq \mathcal{H}^s (E) \). □

Ex: Let \( \mu = \sum_{i=1}^{\infty} \delta_{y_i} \) on \( \mathbb{R} \), \( \mu \) is Borel. Let \( E=(0,1) \), then \( \mu (E) = \infty \). Note that \( \mu (E \cap K) = \infty \), \( \forall K \subset E \) compact. Thus, \( E \) cannot be approximated by compact sets.

However we have:

Theorem 4: \( \mu \) Borel measure on \( \mathbb{R}^n \), \( E \) Borel set, \( \mu (E) < \infty \). Then:

\( \forall \epsilon > 0 \) \( \exists K \subset E \), compact \( \mu (E \cap K) < \epsilon \), In particular:

\[ \mu (E) = \sup \{ \mu (K) : K \subset E, K \text{ compact} \} \]

The previous example shows that \( \mu (E) < \infty \) is needed in Theorem 4.
A locally finite Borel measure \( \mu \) on \( \mathbb{R}^n \) (i.e. \( \mu(K) < \infty \) \( \forall K \subseteq \mathbb{R}^n \) compact) admits outer approximation by open sets.

**Example:** Let \( \mu = \mathcal{H}^1 \) (Borel but not locally finite), measure on \( \mathbb{R}^2 \).

\[
\overline{E} \quad \mu(E) = \mathcal{H}^1(E) < \infty, \quad \text{but} \quad \mathcal{H}^1(A) = \infty \quad \forall A \subseteq \mathbb{R}^2 \text{ open}
\]

However, we have:

**Theorem 2:** \( \mu \) locally finite Borel measure on \( \mathbb{R}^n \), \( E \) Borel set. Then

\[
\mu(E) = \inf \{ \mu(A) : E \subseteq A, A \text{ open} \} = \sup \{ \mu(K) : K \subseteq E, K \text{ compact} \}.
\]

**Radon measure:** A Radon measure \( \mu \) on \( \mathbb{R}^n \) is a Borel regular measure such that \( \mu(K) < \infty \), \( \forall K \subseteq \mathbb{R}^n \), compact. By Theorem 2:

\[
\mu(E) = \inf \{ \mu(A) : E \subseteq A, A \text{ open} \} = \sup \{ \mu(K) : K \subseteq E, K \text{ compact} \},
\]

for every Borel set \( E \).

**Remark:** By Borel regularity, a Radon measure \( \mu \) is characterized on \( M(\mu) \) by its value on compact (or open sets).
Ex: Fix $n$, $0 \leq s \leq n$
- $\mathcal{L}^n$ is Radon measure
- $\mathcal{H}^s$ is not Radon measure (Ex. $\mathcal{H}^1([0,1]^2) = \infty$)
- If $E$ Borel, $\mathcal{H}^s(E) < \infty$, then $\mu = \mathcal{H}^s_{|E}$ is Radon (if $\mu$ is Borel regular on $\mathbb{R}^n$, $E \in \mathcal{M}(\mu)$, $\mu_{|E}$ locally finite $\Rightarrow \mu_{|E}$ is Radon on $\mathbb{R}^n$).

Ex: $\mu = \mathcal{H}^2_{|S}$ on $\mathbb{R}^3$ is Radon.

\[ \text{Def: } \mu_{|E}(F) = \mu(E \cap F) \text{ restriction of a measure.} \]

By Borel regularity we have:

\textbf{Theorem 3: } $\mu$ Radon measure on $\mathbb{R}^n$:
- For every $E \in \mathcal{B}(\mathbb{R}^n)$: $\mu(E) = \inf \{ \mu(A); E \subset A, \text{ open} \}$
- For every $E \in \mathcal{M}(\mu)$: $\mu(E) = \sup \{ \mu(K); K \subset E \text{, compact} \}$

\textbf{Remark: } $\mu, \nu$ Radon, $\mu(K) = \nu(K)$ for all compact $K$ $\Rightarrow$ $\mu = \nu$ on $\mathcal{M}(\mu)$. 
**Push-forward of a measure**

Let $\mu$ be an outer measure on $\mathbb{R}^n$.

Let $f : \mathbb{R}^n \to \mathbb{R}^m$.

The push-forward of $\mu$ through $f$ is the outer measure $f_# \mu$ on $\mathbb{R}^m$ defined by:

$$f_# \mu (E) = \mu (f^{-1}(E)), \quad E \subset \mathbb{R}^m.$$

**Ex:** $f_# \delta_x = \delta_{f(x)}$

Recall, $\delta_x (E) = \begin{cases} 1, & x \in E \\ 0, & \text{otherwise} \end{cases}$

**Prop:** $\mu$ Radon, $f : \mathbb{R}^n \to \mathbb{R}^n$ continuous and proper ( $f^{-1}(\text{compact})$ is compact).

Then $f_# \mu$ is Radon, $\text{supp } f_# \mu = f(\text{supp } \mu)$ and

$$\int_{\mathbb{R}^m} f \, d(f_# \mu) = \int_{\mathbb{R}^n} (\mu \circ f) \, d\mu,$$

$\forall u : \mathbb{R}^n \to [0, +\infty]$ Borel measurable.

**Prop:** $\mu$ Radon, ECIR$^n$ bounded, $\mu(\emptyset) = 0$.

Then, $\forall \epsilon > 0 \exists A$ open, $K$ compact such that $A = E \subset C K$, $\mu(K \setminus A) < \epsilon$. 
Proof: Given $E$, set:

$A_t = \{ x \in E : d(x, \partial E) > t \}$

$K_s = \{ x \in \mathbb{R}^n : d(x, E) \leq s \}$

$\mu(E) = \mu = \bigcup_{t > 0} A_t \Rightarrow \mu(E) = \lim_{t \to 0} \mu(A_t)$ \hspace{1cm} (1)

$E = \bigcap_{s > 0} K_s \Rightarrow \mu(E) = \lim_{s \to 0} \mu(K_s)$ \hspace{1cm} (2)

From (1), for $t$ small enough, set

$A_t = A_t$

Then

$\mu(E \setminus A_t) < \frac{\varepsilon}{2}$, \hspace{0.5cm} for $t$ open, $\bar{A}_t \subset E$

$K_s$ is compact, $E \subset \bar{K}_s$, if we take $s$ small enough, $\mu(K \setminus E) < \frac{\varepsilon}{2}$

Prop. (Foliations by Borel Sets): If $\{ E_t \}_{t \in I}$ is a disjoint family of Borel sets in $\mathbb{R}^n$, and $\mu$ is a Radon measure on $\mathbb{R}^n$, then:

$\{ t : \mu(E_t) > 0 \}$ is at most countable.

Proof: Let $I_k = \{ t \in I : \mu(E_t \cap B_k) > \frac{1}{k^2} \}$

$\Rightarrow \{ t \in I : \mu(E_t) > 0 \} = \bigcup_{k=1}^{\infty} I_k$
\[ \forall J \subset I_k \text{ finite}, \]
\[ \mu(B_k(0)) \geq \mu\left( \bigcup_{t \in J} E_t \cap B_k(0) \right) \]
\[ = \sum_{t \in J} \mu(E_t \cap B_k(0)) \geq \frac{|J|}{k} \]
\[ \therefore \#I_k \leq k \mu(B_k(0)) < \infty \]

Ex. As an application of previous Proposition, a curve of locally finite length can contain at most countably many circular arcs of positive length.

\[ \mathcal{H}'(\Gamma \cap E \{x_0, r\}) > 0 \text{ for at most countably many } r > 0. \]