Theorem: \( n \geq 1 \). Then \( \mathcal{F}(n) \) such that:

If \( \mathcal{F} \) is a family of closed non-degenerate balls on \( \mathbb{R}^n \), and either the set \( \mathcal{C} \) of the centers of the balls in \( \mathcal{F} \) is bounded or

\[
\sup \{ \text{diam} (\overline{B}) : \overline{B} \in \mathcal{F} \} < \infty,
\]

then \( F, F_1, \ldots, F_c(n) \) (possibly empty) subfamilies of \( \mathcal{F} \) such that:

(i) Each \( F_i \) is disjoint and at most countable

(ii) \( \mathcal{C} \subseteq \bigcup_{i=1}^{c(n)} \bigcup_{\overline{B} \in F_i} \overline{B} \)

Note: For every \( x \in \mathbb{R}^n \), the number of balls \( \overline{B} \)

in the collection \( \bigcup_{i=1}^{c(n)} \bigcup_{\overline{B} \in F_i} \overline{B} \)

is less or equal than \( c(n) \).

Idea of Proof:

If \( \overline{B}_1 \) is a ball of "max diameter", get rid of all balls whose centers are in \( \overline{B}_1 \); that is: \( \exists \overline{B}_1 \in \mathcal{F} \) with:

\[
\text{diam} (\overline{B}_1) \geq \frac{2}{3} \sup \{ \text{diam} (\overline{B}) : \overline{B} \in \mathcal{F} \}.
\]
Let $\overline{B}_2$ be any ball from $\mathcal{F}$ whose center does not lie in $\overline{B}_1$ and such that:

$$\text{diam } \overline{B}_2 \geq \frac{2}{3} \sup \{ \text{diam } (\overline{B}) : \overline{B} \in \mathcal{F}, \text{ the center of } \overline{B} \text{ is not in } \overline{B}_1 \}$$

Let $\overline{B}_3$ be any ball from $\mathcal{F}$ whose center does not lie in $\overline{B}_1 \cup \overline{B}_2$ and such that:

$$\text{diam } \overline{B}_3 \geq \frac{2}{3} \sup \{ \text{diam } (\overline{B}) : \overline{B} \in \mathcal{F}, \text{ and the center of } \overline{B} \text{ is not in } \overline{B}_1 \cup \overline{B}_2 \}$$

We continue like this. If this procedure stops after $K$ steps, then we set $M=K$; otherwise, we set $M=\infty$. Let

$$G = \bigcup_{k=1}^{\infty} \overline{B}_k,$$

where $\overline{B}_k := \overline{B}_k(x_k, r_k)$

By construction, we have that:

$$|x_k - x_i| > r_i ; \quad r_k \leq \frac{3}{2} r_i$$

whenever $1 \leq i < K < M$ (which means that the center $x_k$ is not in any of the balls $\overline{B}_i$).

It is proven in Lemma 5.4 that:

$$\# \{ k : 1 < K < N, \overline{B}_k \cap \overline{B}_N \neq \emptyset \} \leq o(n) \quad \forall N < M$$
Then:
(a) $C$ is covered by $G$
(b) $G$ can be divided into $c(n):=\alpha(n)+1$ subfamilies $\mathcal{F}_i$, where each $\mathcal{F}_i$ is disjoint

The proof of Lemma 5.4 is based on a geometric property of $\mathbb{R}^n$ and arrangements of balls with particular radius that don't contain the centers of the other balls (Figure 1).

**Remark:** Let
\[ F = \left\{ B_k = B_{k+\frac{1}{k}}(ke_1) \right\}_{k=1}^\infty \]

The previous theorem is false for this $F$ (since $\sup (\text{diam } B_k) = \infty$). Indeed, in order to cover the centers $C = \left\{ ke_1 \right\}_{k=1}^\infty$, we need infinitely many balls, but the origin belongs to all of them.
Corollary (Vitali's property): If $\mu$ is a Radon measure on $\mathbb{R}^n$, $\mathcal{F}$ is a family of closed non-degenerate balls whose set of centers $\mathcal{G}$ is bounded and $\mu$-measurable, and, for every $x \in \mathcal{G}$:

$$\inf \{\text{diam}(\overline{B}) : \overline{B} \in \mathcal{F}, \overline{B} \text{ has center in } x\} = 0$$

then $\exists \{\overline{B}_i\}$ countable disjoint subfamily such that:

$$\mu \left( \mathcal{G} \setminus \bigcup_{i=1}^{\infty} \overline{B}_i \right) = 0$$

Idea of Proof:

By Besicovitch's covering theorem:

$\exists \mathcal{F}_1, \ldots, \mathcal{F}_{c(n)}$ such that:

$$\mu(\mathcal{G}) \leq \sum_{i=1}^{c(n)} \sum_{\overline{B} \in \mathcal{F}_i} \mu(\mathcal{G} \cap \overline{B})$$

$$\overline{B}_1, \overline{B}_2, \ldots \quad i = 1$$

$$\overline{B}_1, \overline{B}_2, \ldots \quad i = 2$$

$$\overline{B}_1, \overline{B}_2, \ldots \quad i = c(n)$$

$$\overline{B}_1, \overline{B}_2, \ldots \quad i = c(n)$$

$\Rightarrow \exists i \in \{1, \ldots, c(n)\}$ that maximizes $\sum_{\overline{B} \in \mathcal{F}_i} \mu(\mathcal{G} \cap \overline{B})$

Let $G = \mathcal{F}_i$, and thus,

$$\mu(\mathcal{G}) \leq c(n) \sum_{i=1}^{\infty} \mu(\mathcal{G} \cap \overline{B}_i), \overline{B}_i \in G$$
\[ \Rightarrow \mu(B) \leq c(n) \mu(\bigcap \{B : \overline{B_i \cap \bigcup \overline{B_j}} \}) \]

since the balls in the collection G are disjoint.

Now, \exists N_1 such that (since \( \mu(B) < \infty \)):

\[ \mu(\overline{B} \cap \bigcup_{i=1}^{N_1} \overline{B_i}) \geq \frac{\mu(B)}{2c(n)} \]

\[ \Rightarrow \mu(\overline{B} \setminus \bigcup_{i=1}^{N_1} \overline{B_i}) \leq \theta \mu(B) \quad \theta = 1 - \frac{1}{2c(n)} \]

Then \underline{Iterate} to obtain a sequence \( \{N_k\}_{k=1}^{\infty} \) and a sequence \( B_k \subset \mathbb{B} \) and a countable family of closed balls \( \{\overline{B_j}\}_{j=1}^{\infty} \subset \mathbb{F} \) with

\[ t_k = \overline{B} \setminus \bigcup_{j=1}^{N_k} \overline{B_j}, \quad \mu(t_k) \leq \theta^k \mu(B) \]

Since \( \mu(B) < \infty \) \( \Rightarrow \mu(\overline{B} \setminus \bigcup_{j=1}^{\infty} \overline{B_j}) = 0 \).
Lebesgue–Besicovitch Differentiation Theorem:

\[ \mu, \nu \text{ Radon measures on } \mathbb{R}^n. \]

**Upper density:**

\[ D^+_{\mu} \nu(x) = \limsup_{r \to 0} \frac{\nu(B(x, r))}{\mu(B(x, r))}, \forall x \in \text{Supp}(\mu) \]

**Lower density:**

\[ D^-_{\mu} \nu(x) = \liminf_{r \to 0} \frac{\nu(B(x, r))}{\mu(B(x, r))} \]

If \( D^+_{\mu} \nu(x) = D^-_{\mu} \nu(x) \Rightarrow D_{\mu} \nu(x) := D^+_{\mu} \nu(x) \)

\( D_{\mu} \nu(x) \) is the density at \( x \) of \( \nu \) w.r.t. \( \mu \)

**Remark:** Recall a previous remark on "Foliations of Borel sets". Thus, for every \( x \in \mathbb{R}^n \) \( \exists \) at most countable many values of \( r > 0 \) such that either \( \mu(\partial B(x, r)) > 0 \) or \( \nu(\partial B(x, r)) > 0 \). Thus, if \( D_{\mu} \nu \) is defined at \( x \), we have

\[ D_{\mu} \nu(x) = \lim_{r \to 0^+} \frac{\nu(B(x, r))}{\mu(B(x, r))} \]

So in evaluating \( D_{\mu} \nu \) we can use open or closed balls. In the next theorem, closed balls are used in \( D_{\mu} \nu(x) \) because its proof will use Vitali's property; and in the proof of Vitali's property, we can not use open balls instead of closed balls (see book by Ambrosio-Nicola-Fusco, Example 2.20).
Theorem: \( \mu, \nu \) Radon measures on \( \mathbb{R}^n \).

Then:

- \( D_{\mu \nu} \) exists (finite) \( \mu \)-a.e.
- \( D_{\mu \nu} \) is a Borel function, \( D_{\mu \nu} \in L_{loc}^1(\mu) \).

Moreover, \( \nu = (D_{\mu \nu} \mu + \nu^+_{\mu}) \), and \( \nu^+_{\mu} \perp \mu \)

\[ \nu^+_{\mu} \text{ is concentrated on } (\mathbb{R}^n \backslash \text{supp} \mu) \cup \{ x \in \text{supp} \mu : D_{\mu \nu}^+ \nu = \infty \} \]

Idea of Proof: Uses Besicovitch covering theorem. For example, let us prove that \( D_{\mu \nu}(x) \) is finite for \( \mu \)-a.e. \( x \in \mathbb{R}^n \).

Claim: \( t \in (0, \infty) \), \( E \) bounded Borel set in \( \mathbb{R}^n \), then

\[ EC \{ D_{\mu \nu} \leq t \} \Rightarrow \nu(E) \leq t \mu(E) \] (1)

\[ EC \{ D_{\mu \nu} > t \} \Rightarrow \nu(E) > t \mu(E) \] (2)

Fix \( \varepsilon > 0 \), \( \exists A \subset E, \mu(A) \leq \mu(E) + \varepsilon \). For (1), if \( EC \{ D_{\mu \nu} \leq t \} \), then

\[ \mathcal{F} = \{ \overline{B}(x,r) : x \in E, \overline{B}(x,r) \subset A, \nu(\overline{B}(x,r)) \leq (t+\varepsilon) \mu(\overline{B}(x,r)) \} \]

\( \mathcal{F} \) satisfies the assumptions of Vitali's property \( \Rightarrow \exists \{ \overline{B}(x_k,r_k) \} \) a countable disjoint subfamily such that:

\[ \nu(E \setminus \bigcup_{k=1}^{\infty} \overline{B}(x_k,r_k)) = 0, \quad \text{and} \]

\[ \nu(E) = \sum_{k=1}^{\infty} \nu(\overline{B}(x_k,r_k)) \leq (t+\varepsilon) \sum_{k=1}^{\infty} \mu(\overline{B}(x_k,r_k)) \]

\[ \leq (t+\varepsilon) \mu(A) \leq (t+\varepsilon)(\mu(E) + \varepsilon) \]

\( \varepsilon \to 0 \Rightarrow \nu(E) \leq t \mu(E) \). Similar proof for (2).
Now, let $Z = \{ D_{\mu}^+ v = \infty \}$, $Z_{p, q} = \{ D_{\mu}^- v < p < D_{\mu}^+ v \}$, $p, q \in \mathbb{Q}$.

We need to show that $\mu(Z) = \mu(Z_{p, q}) = 0$.

Indeed, $Z \subset \{ D_{\mu}^+ v \geq t \}$, $\forall t > 0$, and thus,

$$\mu(Z \cap B_R) \geq \frac{1}{t} \nu(Z \cap B_R) \leq \frac{\nu(B_R)}{t}$$

$t \to \infty$, $R \to \infty$, $\Rightarrow \mu(Z) = 0$.

Similarly, $\mu(Z_{p, q}) = 0 \Rightarrow D_{\mu} v$ is finite $\mu$-a.e.,

**Lebesgue points**

*Lebesgue points theorem*: $\mu$ Radon on $\mathbb{R}^n$, $p \in [1, \infty)$, $u \in L^p_{\text{loc}}(\mathbb{R}^n, \mu)$, then for $\mu$-a.e. $x \in \mathbb{R}^n$,

$$\lim_{r \to 0^+} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |u(x) - u(y)|^p \, d\mu(y) = 0$$

In this case, we say that $x$ is a Lebesgue point of $u$ with respect to $\mu$.

**Definition**: Let $E \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$, define:

$$\Theta_n(E)(x) = \lim_{r \to 0^+} \frac{|E \cap B(x, r)|}{w_n r^n}$$

if the limit exists, $\Theta_n(E)(x)$ is "the $n$-dimensional density of $E$ at $x"$.
Remark: Let $E \subset \mathbb{R}^n$ Lebesgue measurable, let $\mathcal{M} = \mathbb{L}^n$. Then, by Lebesgue theorem:

$$\lim_{r \to 0^+} \frac{\int_{B(x,r)} \chi_E \, d\mathbb{L}^n}{|B(x,r)|} = \chi_E(x), \text{ for } \mathbb{L}^n\text{-a.e. } x$$

\((*)\) \Rightarrow \frac{|E \cap B(x,r)|}{|B(x,r)|} = \begin{cases} 1, & \text{for } \mathbb{L}^n\text{-a.e. } x \in E \\ 0, & \text{for } \mathbb{L}^n\text{-a.e. } x \in \mathbb{R}^n \setminus E \end{cases}

Def: Given $t \in [0,1]$, the set of points of density $t$ of $E$ is defined as:

$$E(t) = \{ x \in \mathbb{R}^n : \Theta_t(E)(x) = t \}.$$

Thus we have, by \((*)\):

$$|E \Delta E(t)| = 0, \ |(\mathbb{R}^n \setminus E) \Delta E(0)| = 0.$$

"Every Lebesgue measurable set is Lebesgue equivalent to the set of its points of density one".

Proof of Lebesgue points theorem:

Let $\nu = u \mu$ (i.e., $\nu(E) = \int_E u \, d\mu$). Thus,
For $\mu$-a.e. $x \in \mathbb{R}^n$,

$$D_\mu \nu (x) = \lim_{r \to 0^+} \frac{\nu (B(x,r))}{\mu (B(x,r))} = \lim_{r \to 0^+} \frac{1}{\mu (B(x,r))} \int_{B(x,r)} u \, d\mu$$

exists.

For every Borel set $E \subseteq \mathbb{R}^n$, since $\nu = (D_\mu \nu) \mu + \nu^S$:

$$\nu (E) = \int_E D_\mu \nu \, d\mu$$

$$\int_E u \, d\mu$$

$$\therefore u = D_\mu \nu \quad \mu\text{-a.e. on } \mathbb{R}^n.$$