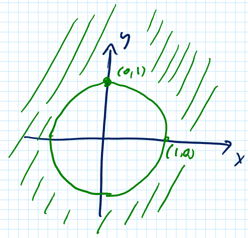
Math 261, Lecture 10, 9/12/18

Today §14.2 (all). Next: §14.3

Recap: functions of several variables

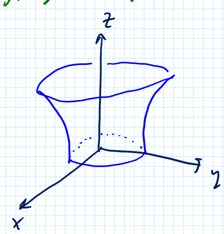
$z = f(x,y) = \sqrt{x^2+y^2-1}$



Domain $x^2+y^2-1 \geq 0$ or $x^2+y^2 \geq 1$

Graph $z = f(x,y)$ or all pts

$(x, y, f(x,y))$ (x,y) in Domain



$z \geq 0$

$z^2 = x^2+y^2-1$

hyperbola of one sheet

Level curves $k = f(x,y)$ solve for (x,y)

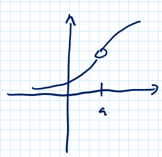
$k = \sqrt{x^2+y^2-1}$

$k^2 = x^2+y^2-1$ or $x^2+y^2 = 1+k^2$

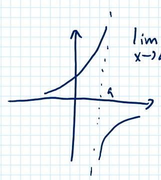
level curves of radius ≥ 1

§ 14.2 Limits and Continuity

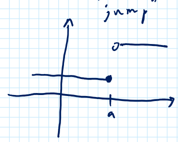
One variable: $\lim_{x \to a} f(x) = L$ exists if $f(x) \approx L$ when $|x-a|$ is small.



$f(x)$ undefined at a but Limit exists

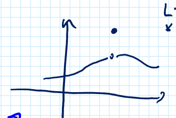


$\lim_{x \to a} f(x)$ DNE



"jump" $\lim_{x \to a} f(x)$ DNE

left and right limits must both exist and be equal.



$\lim_{x \to a} f(x)$ exists $\neq f(a)$

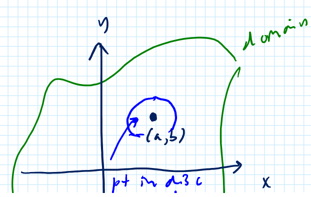
$f(x)$ is continuous at a $\lim_{x \to a} f(x) = f(a)$

If $f(x)$ is continuous at a $\lim_{x \to a} f(x)$ is plug and chug

Defined 2D limits

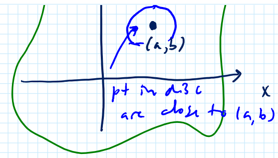
$f(x,y)$

$\lim\limits_{(x,y) \to (a,b)} f(x,y) = L$



$f(x,y)$

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$



$f(x,y) \approx L$

(x,y) close enough to (a,b)

$\sqrt{(x-a)^2 + (y-b)^2}$ is small

or $|x-a| + |y-b|$ is small (small square vs small disk)

$f(x,y)$ is continuous at (a,b) $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$ exists

What functions of two variables are continuous on their domains?

1) polynomials Ex. $f(x,y) = x^2 + 3xy - y^2 + 1$

or $f(x,y) = x^3y^5 - x^2y^4 + 3x^2y - \sqrt{5}\,y$

domain is all (x,y)

$$\lim_{(x,y)\to(2,1)} x^2 + 3xy - y^2 + 1$$

$(2,1)$ in domain, so by continuity limit is

$(2)^2 + 3(2)(1) - (1)^2 + 1 = 10$

2) Compose with one variable function

Ex. $f(x,y) = e^{(x+y)^2}$

or $f(x,y) = \sqrt{1 - x^2 + y^2}$

3) Rational functions Ex. $f(x,y) = \dfrac{xy}{x^2+y^2}$

or $f(x,y) = \dfrac{y}{yx^2 + x^3}$

$f(x,y) = \dfrac{p(x,y)}{q(x,y)}$ domain $q(x,y) \neq 0$

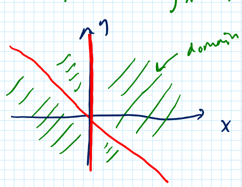
$f(x,y) = \dfrac{y}{yx^2 + x^3}$

$yx^2 + x^3 \neq 0$

$(y+x)x^2 \neq 0$

either $x^2 \neq 0$

or $y \neq -x$



4) Any combination by algebra, composition etc of these examples

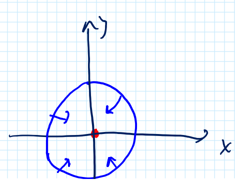
$$f(x,y) = \frac{\sin(2x)\sin(2y)}{xy}$$

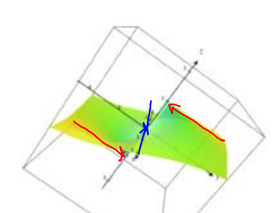
Testing for limits at pts not in domain.

Ex. $f(x,y) = \dfrac{y}{\sqrt{x^2+y^2}}$ domain all $(x,y) \neq (0,0)$

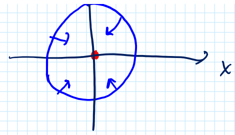
$$\lim_{(x,y)\to(0,0)} \frac{y}{\sqrt{x^2+y^2}}$$ exist?

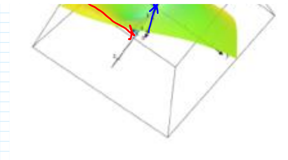
test along axes





$(x,y) \to (0,0) \quad \sqrt{x^2+y^2}$





test along axes

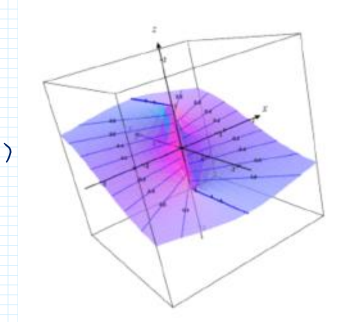
x - axis , $y = 0$

$$\lim_{(x,0)\to(0,0)} \frac{y}{\sqrt{x^2+y^2}} = \lim_{x\to 0} \frac{0}{\sqrt{x^2+0^2}} = 0$$

y - axis , $x = 0$

$$\lim_{(0,y)\to(0,0)} \frac{y}{\sqrt{x^2+y^2}} = \lim_{y\to 0} \frac{y}{\sqrt{0^2+y^2}} = \lim_{y\to 0} \frac{y}{|y|} \text{ DNE}$$

(-1 from "left" +1 from "right")

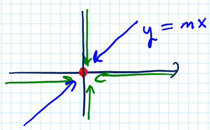


$$\boxed{\text{Conclusion} \quad \lim_{(x,y)\to(0,0)} \frac{y}{\sqrt{x^2+y^2}} \text{ DNE}}$$

Ex. $f(x,y) = \dfrac{xy}{x^2+y^2}$ $\quad \lim_{(x,y)\to(0,0)} f(x,y)$ exists?

along x - axis $\quad \lim_{(x,0)\to(0,0)} \dfrac{xy}{x^2+y^2} = \lim_{x\to 0} \dfrac{x0}{x^2+0^2} = 0$

along y - axis $\quad \lim_{(0,y)\to(0,0)} \dfrac{xy}{x^2+y^2} = \lim_{y\to 0} \dfrac{0y}{0^2+y^2} = 0$

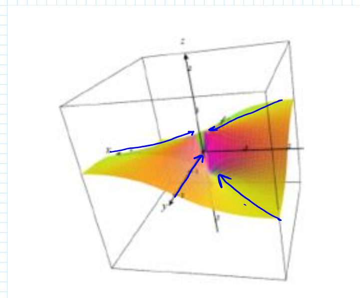


along $y = mx$ $\quad \lim_{(x,mx)\to(0,0)} \dfrac{xy}{x^2+y^2} = \lim_{x\to 0} \dfrac{x(mx)}{x^2+(mx)^2}$

$$= \lim_{x\to 0} \frac{m\cancel{x^2}}{(1+m^2)\cancel{x^2}}$$

$$= \lim_{x\to 0} \frac{m}{1+m^2} \neq 0$$

if $m \neq 0$



$$\boxed{\text{Conclusion} \quad \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \text{ DNE}}$$

Ex. $f(x,y) = \dfrac{y\sin(x)^2}{y^2+x^4}$, $\lim_{(x,0)\to(0,0)} \dfrac{y\sin(x)^2}{y^2+x^4} = \dfrac{0\sin(x)^2}{0^2+x^4} = 0$

$\lim_{(0,y)\to(0,0)} \dfrac{y\sin(0)^2}{y^2+0^4} = 0$

$y = mx$

$$\lim_{(x,mx)\to(0,0)} \frac{y\sin(x)^2}{y^2+x^4} = \lim_{x\to 0} \frac{mx\sin(x)^2}{m^2x^2+x^4}$$

$$= \lim_{x\to 0} \Big(\frac{\sin(x)}{\cancel{x}}\Big)^{2} \frac{m\cancel{x}}{m^2+\cancel{x^2}} = 0 \text{ as well!}$$

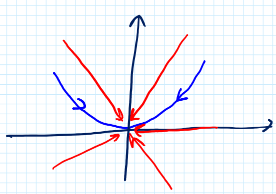
$m \neq 0$

Now try $y = x^2$!

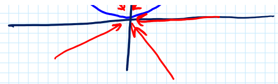
Non zero limit !

$$\lim_{(x,x^2)\to(0,0)} \frac{y\sin(x)^2}{y^2+x^4} = \lim_{x\to 0} \frac{x^2\sin(x)^2}{(x^2)^2+x^4} = \lim_{x\to 0} \frac{\cancel{x^2}\sin(x)^2}{2x^{\cancel{4}\,2}}$$

$$= \lim_{x\to 0} \frac{\sin(x)^2}{2x^2} = \frac{1}{2} \neq 0$$



So approaching by a parabolic path is different than approaching by any linear direction.



path is different than approaching
by any linear direction.

Conclusion: $\boxed{\lim_{(x,y)\to(0,0)} \frac{y\sin(x)^2}{y^2+x^4} \text{ DNE}}$