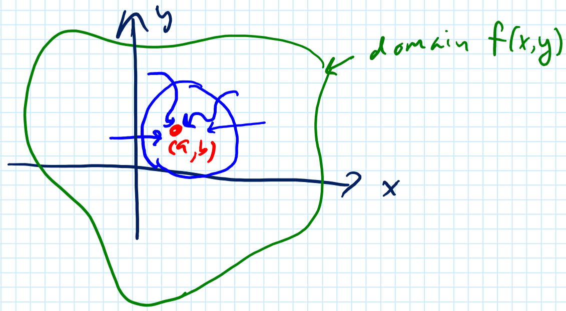
Math 261, Lecture 11, 9/14/18

Today: §14.3, Next: §14.4

Recap: If $\lim_{(x,y)\to(a,b)} f(x,y)$ exists, the limits along any path leading to $(a,b)$ must exist and agree.




Strategy:

(a,b) in domain?

YES → evaluate

NO → Try x-axis, y-axis, $y=mx$, ...
Find two paths with different limits

Fall 2017 Exam 1

**4.** Consider the following two limits:

I. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+xy+y^2}$ II. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+y}{e^x+e^y}$

A. I. does not exist, II. 0
B. I. does not exist, II. does not exist
C. I. 0, II. does not exist
D. I. 0, II. 0
E. I. 0, II. 1

I. (0,0) in domain?
NO, denominator $x^2+xy+y^2$ is 0 at (0,0)
↳ Suspect does not exist

x-axis $\displaystyle\lim_{(x,0)\to(0,0)} \frac{x0}{x^2+x0+0^2} = 0$, $x=0$ y-axis $=0$, Try $y=mx$

$$\lim_{(x,mx)\to(0,0)} \frac{x(mx)}{x^2+x(mx)+(mx)^2} = \lim_{x\to 0} \frac{mx^2}{x^2+mx^2+m^2x^2} = \frac{m}{1+m+m^2}$$

DNE since limit depends on m!

II. $\dfrac{x+y}{e^x+e^y}$ (0,0) domain?
YES

$$\lim_{(x,y)\to(0,0)} \frac{x+y}{e^x+e^y} = \frac{0+0}{e^0+e^0} = \frac{0}{2} = 0$$

## §14.3 Partial Derivatives

$z = f(x,y)$

partial derivative in x

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} f(x,y)$$ "$\partial$ vs $d$"

$$= \lim \quad f(x+h,y) - f(x,y)$$

$$= \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

partial derivative in $y$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Ex. $f(x,y) = x^2 y$ $\quad \frac{\partial}{\partial x} f(x,y)$

$$\frac{\partial}{\partial x} f(x,y) = \lim_{h \to 0} \frac{(x+h)^2 y - x^2 y}{h} = \lim_{h \to 0} \frac{(x^2 + 2hx + h^2) y - x^2 y}{h}$$

$$= \lim_{h \to 0} \frac{2\cancel{h}xy + h^{\cancel{2}} y}{\cancel{h}} = \lim_{h \to 0} 2xy + \cancel{h}^{\,0} y$$

$$= 2xy$$

For $\frac{\partial}{\partial x}$ treat $y$ as a constant, diff as usual.

For $\frac{\partial}{\partial y}$ treat $x$ as a constant, diff as usual (in $y$)

Ex. $f(x,y) = x^3 y^2 - 3x^4 y^3 - x^{3/2} y^{5/2}$

$$f_x = \frac{\partial}{\partial x} f(x,y) = 3x^2 y^2 - 12 x^3 y^3 - \frac{3}{2} x^{1/2} y^{5/2}$$

$$f_y = \frac{\partial}{\partial y} f(x,y) = 2x^3 y - 9x^4 y^2 - \frac{5}{2} x^{3/2} y^{3/2}$$

Ex. $f(x,y) = \frac{xy}{x^2 + y^2}$, Find $f_x$

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)\, g(x) - f(x)\, g'(x)}{[g(x)]^2} \quad \text{(Quotient Rule)}$$

$$\frac{y(x^2+y^2) - xy\cdot\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{y(x^2+y^2) - xy\cdot\left[2x\right]}{(x^2+y^2)^2}$$

$$= \frac{x^2y + y^3 - 2x^2y}{(x^2+y^2)^2} = \frac{-x^2y + y^3}{(x^2+y^2)^2}$$

## Mixing Implicit and Partial Differentiation

Ex. $3x^2y + x^3yz^2 + 4xyz = 5$

NOT of the form $z = f(x,y)$, but want to view $z$ implicitly as a function of $(x,y)$.

Take partial of implicit eqn w.r.t. $x$.

$$\frac{\partial}{\partial x}\left(3x^2y + x^3yz^2 + 4xyz\right) = \frac{\partial}{\partial x}5$$

product rule

$$6xy + \left[3x^2y\cdot z^2 + x^3y\frac{\partial}{\partial x}(z^2)\right] + \left[4y\cdot z + 4xy\frac{\partial}{\partial x}(z)\right] = 0$$

$$6xy + 3x^2yz^2 + x^3y\left(2z\frac{\partial z}{\partial x}\right) + 4yz + 4xy\frac{\partial z}{\partial x} = 0$$

terms not involving $\frac{\partial z}{\partial x}$

$$2x^3yz\frac{\partial z}{\partial x} + 4xy\frac{\partial z}{\partial x} = -6xy - 3x^2yz^2 - 4yz$$

Solve for $\frac{\partial z}{\partial x}$

$$\frac{\partial z}{\partial x} = \frac{-6xy - 3x^2yz^2 - 4yz}{\quad}$$

$2x^3yz + 4xy$

Mixed Partial Derivatives

$y = f(x)$ , $\frac{dy}{dx} = f'(x)$ , $\frac{d^2y}{dx^2} = f''(x)$ , ...

$z = f(x,y)$ $\quad f_{xx} = \frac{\partial^2 f}{\partial x^2}$ derive twice in x

$f_{yy} = \frac{\partial^2 f}{\partial y^2}$ derive twice in y

$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f\right)$ , $f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f\right)$

<u>Clairaut's Theorem</u>. If $f_{xy}$ and $f_{yx}$ are continuous then $f_{xy} = f_{yx}$

Order doesn't matter when taking partials (Mostly! For our purposes, at least)

Ex. $\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} e^{x^2y}\right) = \frac{\partial}{\partial y}\left(e^{x^2y} \frac{\partial}{\partial x}(x^2y)\right)$

$= \frac{\partial}{\partial y}\left(e^{x^2y}\, 2xy\right)$

$= e^{x^2y}\left(\frac{\partial}{\partial y} x^2y\right) \cdot 2xy + e^{x^2y} \cdot \frac{\partial}{\partial y}(2xy)$

$= e^{x^2y} \cdot x^2 \cdot 2xy + e^{x^2y} \cdot 2x$

$= \boxed{(2x^3y + 2x)e^{x^2y}}$

$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} e^{x^2y}\right) = \frac{\partial}{\partial x}\left(e^{x^2y}\, x^2\right)$

$= \left(\frac{\partial}{\partial x} e^{x^2y}\right) x^2 + e^{x^2y}\, 2x$

These are equal!

$$= \left(\frac{\partial}{\partial x} e^{x^2 y}\right) x^2 + e^{x^2 y}\, 2x$$

These are
equal!

$$= \left(e^{x^2 y}\, 2xy\right) x^2 + e^{x^2 y}\, 2x$$

$$= \boxed{\left(2x^3 y + 2x\right) e^{x^2 y}}$$