Math 261, Lecture 12, 9/17/18

Today: §14.4 (except Ex. 5,6), Next: §14.4 (Ex. 5,6), §14.5

Recap. $z = f(x,y)$

partial derivative in x $\quad \frac{\partial z}{\partial x} = \frac{\partial}{\partial x} f(x,y) = f_x(x,y)$

$$= \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

partial derivative in y $\quad \frac{\partial z}{\partial y} = \frac{\partial}{\partial y} f(x,y) = f_y(x,y)$

$$= \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

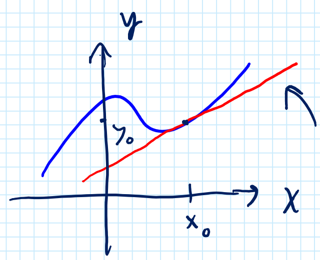
* Rule - Hold the other variable constant, then differentiate as normal in the chosen variable.

All the rules of usual (single variable) differentiation apply still.

§14.4 Tangent Planes and Linear Approximations

Review of single variable calc

$y = f(x)$



$f'(x_0)$ is slope of tangent line

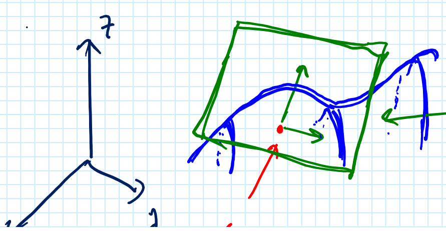
eqn of tangent line:

$$(y - y_0) = f'(x_0)(x - x_0)$$

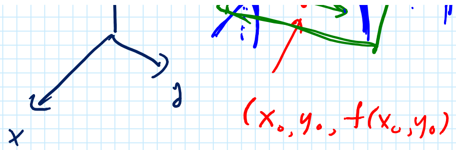
linear approximation $\quad \Delta y \approx f'(x_0)\,\Delta x$

if Δx is "small".

In two variables $z = f(x,y)$



tangent plane



$(x_0, y_0, f(x_0,y_0)) = (x_0, y_0, z_0)$

Find the tangent plane at (x_0, y_0, z_0), $z_0 = f(x_0,y_0)$

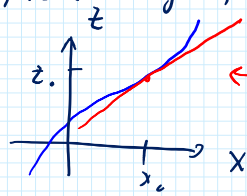
General eq'n of plane containing (x_0,y_0,z_0): $A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$

↳ Solve for $z - z_0$

$$C(z-z_0) = -A(x-x_0) - B(y-y_0)$$

$$(z-z_0) = -\frac{A}{C}(x-x_0) - \frac{B}{C}(y-y_0)$$

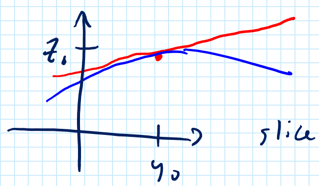
Slice Method $y = y_0$ $\quad z - z_0 = -\frac{A}{C}(x-x_0)$



← $-\frac{A}{C}$ is slope of this tangent line!

$y = y_0$

$$-\frac{A}{C} = f_x(x_0,y_0)$$



← $-\frac{B}{C} = f_y(x_0,y_0)$

slice $x = x_0$

The partial derivative is the derivative of the curve along the slice.

$$\boxed{\text{Formula for tangent plane: } \quad z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}$$

We say that f is differentiable at (x_0,y_0) if $\Delta z \approx f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y$

"Good tangent plane approximation to surface"

✗ Warning! f_x, f_y may exist but f may not be differentiable

Good News! If f_x, f_y are continuous No Problem!

$\hookrightarrow \lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists !!

Ex. $f(x,y) = \dfrac{xy}{x^2+y^2}$ $\quad \lim_{(x,y)\to(0,0)} f(x,y)$ DNE.

so one of f_x or f_y must be discontinuous at $(0,0)$

$f_x = \dfrac{-x^2y - y^3}{(x^2+y^2)^2}$ is not continuous at $(0,0)$

Ex. $f(x,y) = e^{x^2y}$ Find tangent plane at $(1,0)$

$x_0 = 1,\ y_0 = 0,\ z_0 = e^{x_0^2 y_0} = e^{1^2\cdot 0} = e^0 = 1$

base point of tangent plane is $(1,0,1)$

$$f_x = e^{x^2y}\frac{\partial}{\partial x}x^2y = e^{x^2y}\,2xy$$

$$f_y = e^{x^2y}\frac{\partial}{\partial y}x^2y = e^{x^2y}\,x^2$$

$f_x(1,0) = e^{1^2\cdot 0}\cdot 2\cdot 1\cdot 0 = 0$

$f_y(1,0) = e^{1^2\cdot 0}\cdot 1^2 = 1$

Answer: $z-1 = 0(x-1) + 1(y-0)$

or $z-1 = y$

Ex. Use calculus to show that $\arctan(xy)$ has a limit at $(0,0)$

$$\frac{\partial}{\partial x}\arctan(xy) = \frac{1}{1+(xy)^2}\cdot y = \frac{y}{1+x^2y^2}$$

$$\frac{\partial}{\partial y}\arctan(xy) = \frac{1}{1+(xy)^2}\cdot x = \frac{x}{1+x^2y^2}$$

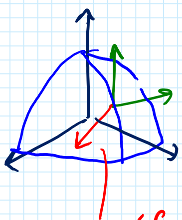
$(0,0)$ is in the domain of both of the rational functions f_x and f_y, so both are continuous and $\lim_{(x,y)\to(0,0)} \arctan(xy)$ exists.

Remark. Eqn of tangent plane

$$0 = f_x(x-x_0) + f_y(y-y_0) - (z-z_0)$$

Normal to tangent plane is $(f_x, f_y, -1)$

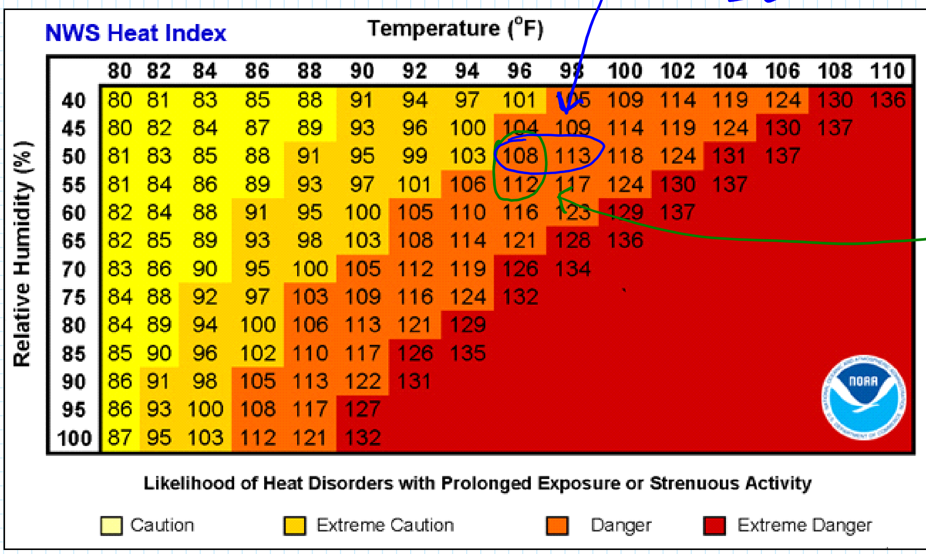
gradient vector to surface.



$(f_x, f_y, -1)$

$F(t,h)$

$\frac{\partial F}{\partial t} \approx \frac{\Delta F}{\Delta t}$



NWS Heat Index — Temperature (°F)

Relative Humidity (%)

	80	82	84	86	88	90	92	94	96	98	100	102	104	106	108	110
40	80	81	83	85	88	91	94	97	101	105	109	114	119	124	130	136
45	80	82	84	87	89	93	96	100	104	109	114	119	124	130	137	
50	81	83	85	88	91	95	99	103	108	113	118	124	131	137		
55	81	84	86	89	93	97	101	106	112	117	124	130	137			
60	82	84	88	91	95	100	105	110	116	123	129	137				
65	82	85	89	93	98	103	108	114	121	128	136					
70	83	86	90	95	100	105	112	119	126	134						
75	84	88	92	97	103	109	116	124	132							
80	84	89	94	100	106	113	121	129								
85	85	90	96	102	110	117	126	135								
90	86	91	98	105	113	122	131									
95	86	93	100	108	117	127										
100	87	95	103	112	121	132										

Likelihood of Heat Disorders with Prolonged Exposure or Strenuous Activity

Caution | Extreme Caution | Danger | Extreme Danger

$\frac{\partial F}{\partial h} \approx \frac{\Delta F}{\Delta h}$

* Heat Index Chart from National Weather Service, weather.gov/safety/heat-index

Ex. 97°F and 51% humidity. Estimate heat index from table.

Closest point on table $(96°F, 50\%)$, $\;96 = t_0$, $50 = h_0$ $\qquad z_0 = F(96,50) = 108°F$

$$\overset{z = F(t,h)}{F(97,51)} \approx \overset{z_0 = F(t_0,h_0)}{F(t_0,h_0)} + F_t(96,50)\,\Delta t + F_h(96,50)\,\Delta h$$

$\Delta t = 97 - 96 = 1°F$ $\qquad F_t(96,50) \approx \frac{\Delta F}{\Delta t} = \frac{113-108}{98-96}$

$\Delta h = 51 - 50 = 1\%$ $\qquad F_h(96,50) \approx \frac{\Delta F}{\Delta h} = \frac{112-108}{55-50} = \frac{4}{5}$

$\Delta h = 51 - 50 = 1\%$

$$F_h(96, 50) \approx \frac{\Delta F}{\Delta h} = \frac{112 - 108}{55 - 50} = \frac{4}{5}$$

Answer $F(97, 51) \approx 108 + \frac{5}{2} \cdot 1 + \frac{4}{5} \cdot 1$

$= 111.3\,^\circ F$