Math 261, Lecture 13, 9/19/18

Today: §14.4 Ex. 5, 6, §14.5, Next: §14.5 (final), §14.6 (begin)

Recap: $\underbrace{z = f(x,y)}_{\text{surface}}$ tangent plane at (x_0, y_0, z_0) $z_0 = f(x_0, y_0)$

$$z - z_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$$

Differentiable

$$\Delta z \approx f_x \cdot \Delta x + f_y \cdot \Delta y$$

tangent plane approximates surface well.

§14.4, Differentials

$$z = f(x,y)$$

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

"infinitesimal change in z"

$$w = f(x,y,z)$$

$$dw = f_x\, dx + f_y\, dy + f_z\, dz$$

Ex. Rectangular box $\underset{x}{10\text{cm}} \times \underset{y}{30\text{cm}} \times \underset{z}{12\text{cm}}$

made up to .1 cm error in dimensional measurements

Estimate volume error

$$V = xyz$$

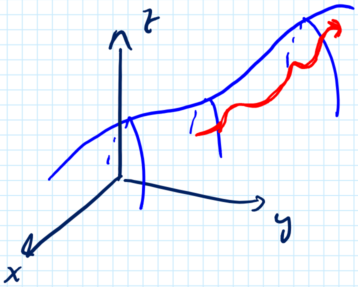
$$dV = yz\, dx + xz\, dy + xy\, dz$$

error in volume

errors in dimensional measurements

$$dV = 30\cdot 12\,dx + 10\cdot 12\,dy + 10\cdot 30\,dz$$
$$= 30\cdot 12\cdot (0.1) + 10\cdot 12\,(0.1) + 10\cdot 30\,(0.1)$$
$$= 36 + 12 + 30 = 78 \text{ cm}^3$$

§14.5 Chain Rule



$$z = f(x,y)$$

$$x = x(t)$$
$$y = y(t)$$

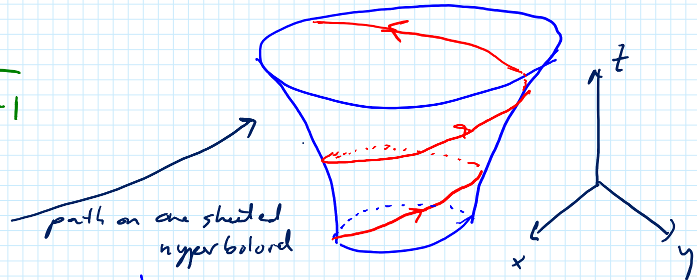
and $z = z(t) = f(x(t), y(t))$

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$ gives a path on the surface $z = f(x,y)$

Find $\dfrac{d\vec{r}}{dt} = \left\langle \dfrac{dx}{dt}, \dfrac{dy}{dt}, \dfrac{dz}{dt} \right\rangle$ How to differentiate?

Chain Rule
$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$
$$= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Ex. $z = \sqrt{x^2+y^2-1}$



$$x = t\cos t$$
$$y = t\sin t$$

$$\frac{dz}{dt} = \,?$$

$$f(x,y) = \sqrt{x^2+y^2-1}$$

$$z = \sqrt{t^2\cos(t)^2 + t^2\sin(t)^2 - 1}$$
$$= \sqrt{t^2(\underbrace{\cos(t)^2+\sin(t)^2}_{1}) - 1}$$
$$= \sqrt{t^2-1}$$

* substituting x, y for expressions in t

$f(x,y) = \sqrt{\;\;}$

$= \sqrt{t^2-1}$ * substituting x, y for expressions in t

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$\frac{\partial f}{\partial x} = \frac{1}{\cancel{2}\sqrt{x^2+y^2-1}} \cdot \cancel{2}x$$

$$\frac{\partial f}{\partial y} = \frac{1}{\cancel{2}\sqrt{x^2+y^2-1}} \cancel{2}y$$

$$\frac{dx}{dt} = \frac{d}{dt}[t\cos t] = \cos t - t\sin t$$

$$\frac{dy}{dt} = \frac{d}{dt}[t\sin t] = \sin t + t\cos t$$

All together (in t)

$$\frac{dz}{dt} = \underbrace{\frac{t\cos t}{\sqrt{t^2-1}}}_{\frac{\partial f}{\partial x}} \cdot \underbrace{\left[\cos t - t\sin t\right]}_{\frac{dx}{dt}} + \underbrace{\frac{t\sin t}{\sqrt{t^2-1}}}_{\frac{\partial f}{\partial y}} \cdot \underbrace{\left[\sin t + t\cos t\right]}_{\frac{dy}{dt}}$$

* substituting $x = t\cos t$, $y = t\sin t$

$$= \frac{t\cos(t)^2 - t^2\cancel{\cos(t)\sin(t)} + t\sin(t)^2 + t^2\cancel{\sin(t)\cos(t)}}{\sqrt{t^2-1}}$$

$$= \frac{t\,(\underbrace{\cos(t)^2 + \sin(t)^2}_{1})}{\sqrt{t^2-1}} = \frac{t}{\sqrt{t^2-1}}$$

Now,

$z = f(x,y) \qquad x = x(s,t)$

$\qquad\qquad\qquad\; y = y(s,t)$

Find $\dfrac{\partial z}{\partial s}, \dfrac{\partial z}{\partial t}$

Chain Rule, Part II

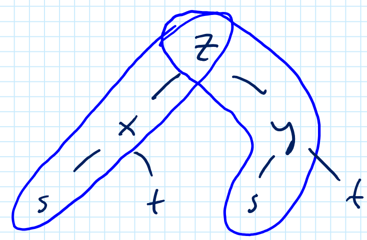
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} \qquad \Bigg| \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

Chain Rule;

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$
$$= \frac{\partial f}{\partial x}\,\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\,\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$
$$= \frac{\partial f}{\partial x}\,\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\,\frac{\partial y}{\partial t}$$

Tree of variables



For $\frac{\partial z}{\partial s}$ follow all paths down that end at s.

Ex. $z = \sin(x+y)$
$x = x(s,t) = se^t$
$y = y(s,t) = s^3 t$

Find $\frac{\partial z}{\partial s}$, $\frac{\partial z}{\partial t}$

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial x} = \cos(x+y)\cdot 1 = \cos(x+y)$$

$$\frac{\partial f}{\partial y} = \cos(x+y)\cdot 1 = \cos(x+y)$$

$$\frac{\partial x}{\partial s} = e^t \qquad \frac{\partial y}{\partial s} = 3s^2 t$$

$$\frac{\partial z}{\partial s} = \cos(se^t + s^3 t)\cdot e^t + \cos(se^t + s^3 t)\cdot 3s^2 t$$
$$= (e^t + 3s^2 t)\cos(se^t + s^3 t)$$

$\frac{\partial z}{\partial t} = ?$ $\qquad \frac{\partial x}{\partial t} = se^t$, $\frac{\partial y}{\partial t} = s^3$

$$\frac{\partial z}{\partial t} = \cos(se^t + s^3 t) \cdot se^t + \cos(se^t + s^3 t)\, s^3$$

$$= (se^t + s^3)\cos(se^t + s^3 t)$$

If I substitute $x = se^t$ and $y = s^3 t$ into

$z = \sin(x+y)$, I get

$z = \sin(se^t + s^3 t)$.

Now, taking the partials in s and t, I get the same answer!