Math 261, Lecture 14

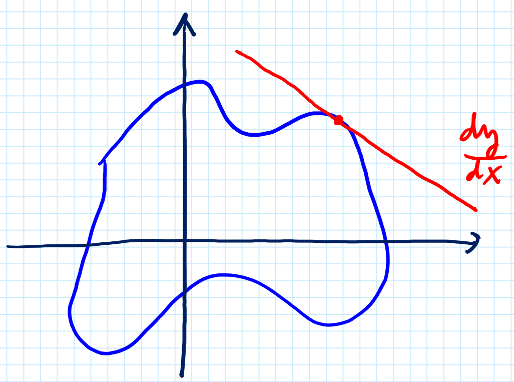
* These notes are not based on a lecture, as there was a guest lecturer on this day.

Topics §14.5 (finish) §14.6 (begin)

§14.5 Implicit Differentiation

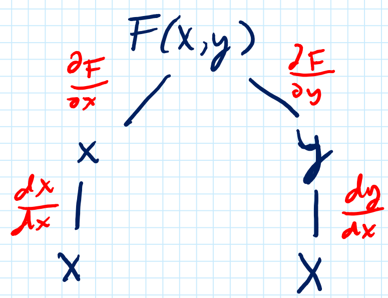
Let $F(x,y) = c$ be an <u>implicit function</u> in x, so

$F(x,y)$ defines a curve in the xy-plane



How do we find the slope of the tangent line?

Let's apply the Multivariable Chain Rule



$$\frac{dF}{dx} = \frac{\partial F}{\partial x} \cdot \cancelto{1}{\frac{dx}{dx}} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx}$$

Need to solve for this

We have that $F(x,y) = c$, so

We have that $F(x,y) = c$, so

$$\frac{dF}{dx} = \frac{d}{dx}c = 0 \text{ since } c \text{ is a constant.}$$

All together,

$$0 = \frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx}$$

or $\quad -F_x = F_y \frac{dy}{dx}$

or $\quad \boxed{\dfrac{dy}{dx} = \dfrac{-F_x}{F_y}}$ Implicit function theorem.

Ex. Let $F(x,y) = ye^{xy} - x$

Find $\frac{dy}{dx}$ at $(0,2)$ for the curve $F(x,y) = 2$

* Check that $F(0,2) = 2e^{0\cdot 2} - 0 = 2$ so $(0,2)$ is on the curve.

Now $\dfrac{dy}{dx} = \dfrac{-F_x}{F_y}$ by Implicit Function Thm

$$F_x = y^2e^{xy} - 1$$

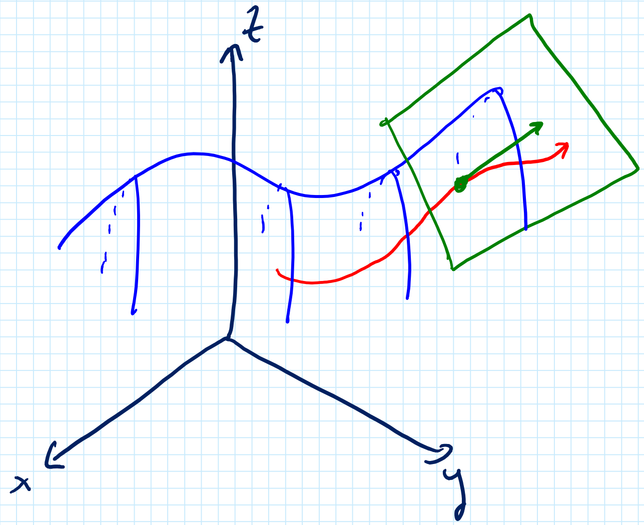
$$F_y = e^{xy} + xye^{xy} + 0$$

So $F_x(0,2) = (2)^2e^{0\cdot 2} - 1 = 4 - 1 = 3$

$F_y(0,2) = e^{0\cdot 2} - 0\cdot 2e^{0\cdot 2} = 1$

Answer $\boxed{\frac{dy}{dx} = \frac{-F_x(0,2)}{F_y(0,2)} = \frac{-3}{1} = -3}$

§14.6 Directional Derivatives and the Gradient Vector



We have a curve on a surface passing thru a point, at what rate is z changing as the particle passes the point?

If the surface is $z = f(x,y)$
curve is $f(x(t), y(t))$ for some pair $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ of functions on t.

By the Chain Rule $\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$

This depends on the parametrization! So let's instead require that the particle approaches with unit speed, that is $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 1$.

This motivates the concept of a Directional Derivative

Let $\vec{u} = \langle a, b \rangle$ be a unit vector, $|\vec{u}| = \sqrt{a^2+b^2} = 1$

$z = f(x,y)$ the derivative in the direction of $\vec{u}$ is defined to be

$$D_{\vec{u}} f(x,y) = \lim_{h \to 0} \frac{f(x+ah, y+bh) - f(x,y)}{h}$$

$$\boxed{\text{If } f \text{ is differentiable, then } D_{\vec{u}} f = \frac{\partial f}{\partial x} \cdot a + \frac{\partial f}{\partial y} \cdot b}$$

Key Fact

* The directional derivative is a number not a vector.

Ex. Find the derivative of $f(x,y) = \sin(3x - 5y)$ at $(0,0)$ in the direction of $\langle -1, 1 \rangle$

$\vec{v} = \langle -1, 1 \rangle$, the $\dfrac{\vec{v}}{|\vec{v}|} = \dfrac{\langle -1, 1 \rangle}{\sqrt{(-1)^2 + 1^2}} = \left\langle \dfrac{-1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}} \right\rangle = \vec{u}$

is the direction of $\vec{v}$.

$$\frac{\partial f}{\partial x} = \cos(3x - 5y) \cdot 3$$

$\dfrac{\partial f}{\partial x}(0,0) = \cos(3 \cdot 0 - 5 \cdot 0) \cdot 3$
$= 1 \cdot 3 = 3$

$$\frac{\partial x}{} = \qquad\qquad \frac{\partial}{\partial x} \qquad = 1\cdot 3 = 3$$

$$\frac{\partial f}{\partial y} = \cos(3x - 5y)\cdot -5 \qquad \frac{\partial f}{\partial y}(0,0) = \cos(3\cdot 0 - 5\cdot 0)(-5) = -5$$

$$\boxed{\begin{aligned} \text{So } D_{\vec{u}} f(0,0) &= 3\cdot \frac{-1}{\sqrt{2}} - 5\cdot\frac{1}{\sqrt{2}} \\ &= -\frac{8}{\sqrt{2}} = -4\sqrt{2} \end{aligned}}$$

We define the <u>Gradient</u> of $f(x,y)$ to be

$$\vec{\nabla} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

Thus if $\vec{u} = \langle a, b\rangle$ is a unit vector

$$\boxed{D_{\vec{u}} f = \frac{\partial f}{\partial x}\cdot a + \frac{\partial f}{\partial y}\cdot b = \vec{\nabla} f \cdot \vec{u}}$$

Ex. $f(x,y) = \sqrt{4 - x^3 - y^3 + 3xy^2}$ Find $\vec{\nabla} f$

$$\frac{\partial f}{\partial x} = \frac{1}{2\sqrt{4 - x^3 - y^3 + 3xy^2}}\cdot(-3x^2 + 3y^2)$$

$$= \frac{3}{2}\cdot\frac{y^2 - x^2}{\sqrt{4 - x^3 - y^3 + 3xy^2}}$$

$$\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{4-x^3-y^3+3xy^2}} \cdot \left(-3y^2+6xy\right)$$

$$= \frac{3}{2} \frac{y(2x-y)}{\sqrt{4-x^3-y^3+3xy^2}}$$

Answer

$$\vec{\nabla} f(x,y) = \left\langle \frac{3}{2} \frac{y^2-x^2}{\sqrt{4-x^3-y^3+3xy^2}}, \frac{3}{2} \frac{y(2x-y)}{\sqrt{4-x^3-y^3+3xy^2}} \right\rangle$$

Works the same in 3 variables

$w = f(x,y,z)$, $\quad \vec{u} = \langle a,b,c \rangle \quad |\vec{u}| = \sqrt{a^2+b^2+c^2} = 1$

$$\vec{\nabla} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

$$D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u}$$

Ex. Find the derivative of $f(x,y,z) = x^3 - 2y^2z + xyz$ in the direction of $\vec{v} = \langle 2,3,6 \rangle$ at $(1,2,-1)$

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 2,3,6 \rangle}{\sqrt{2^2+3^2+6^2}} = \left\langle \frac{2}{7}, \frac{3}{7}, \frac{6}{7} \right\rangle$$

$f_x = 3x^2 + yz \qquad f_x(1,2,-1) = 3\cdot1^2 + 2\cdot(-1) = 1$

$$f_y = -4yz + xz \qquad f_y(1,2,-1) = -4(2)(-1) + 1(-1) = 3$$

$$f_z = -2y^2 + xy \qquad f_z(1,2,-1) = -2(2)^2 + 1(2)$$

$$= -6$$

Answer

$$D_{\vec{u}} f(1,2,-1) = 1 \cdot \frac{2}{7} + 3 \cdot \frac{3}{7} - 6 \cdot \frac{6}{7}$$

$$= -\frac{25}{7}$$