Math 261, Lecture 15, 9/24/18

Announcements

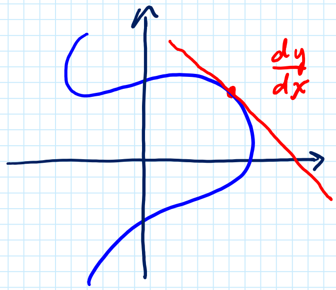
- EXAM 1 is Tuesday, OCT 2, at 8:00 pm in Elliott Hall.
- Lessons 2-16 are testable.
- Past Exams Archive, www.math.purdue.edu/ma261

Today: 14.6 (finish), Next: 14.7 (beginning - Ex. 4)

Recap: $F(x,y) = c$ level curve

$$\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y}}$$

"Implicit Function Theorem"



For $z = f(x,y)$ the gradient $\vec{\nabla} f = \langle f_x, f_y \rangle$

$w = f(x,y,z)$ " ——— " $\vec{\nabla} f = \langle f_x, f_y, f_z \rangle$

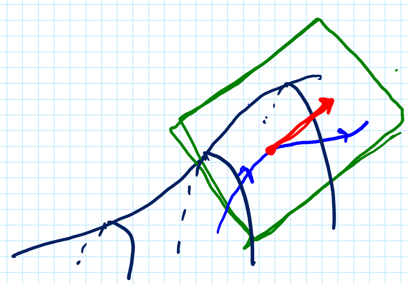
↳ Is a vector valued function

Directional derivative $|\vec{u}| = 1$ then $\vec{u} = \langle a, b \rangle$

$$D_{\vec{u}} f(x_0,y_0) = f_x(x_0,y_0)a + f_y(x_0,y_0)b$$
$$= \vec{\nabla} f \cdot \vec{u}$$

this is a number

"speed of a particle moving on a path in surface"



9. The directional derivative of $f(x, y, z) = y^2 e^{x-z}$ at the point $(3, 1, 2)$ in the direction $2\vec{i} + 5\vec{j} + \vec{k}$ is:

A. $\frac{13e}{\sqrt{30}}$

B. $13e$

C. $\frac{35e}{\sqrt{14}}$

D. $\frac{11e}{\sqrt{30}}$

E. $11e$

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$\vec{v} = \langle 2, 5, 1 \rangle$ direction is $\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 2,5,1 \rangle}{\sqrt{2^2+5^2+1^2}} = \frac{1}{\sqrt{30}} \langle 2,5,1 \rangle$

$f(x,y,z) = y^2 e^{x-z}$

$f_x = y^2 e^{x-z}$

$f_y = 2y e^{x-z}$ at $(3,1,2)$

$f_z = -y^2 e^{x-z}$

$f_x(3,1,2) = 1^2 e^{3-2} = e$

$f_y(3,1,2) = 2 \cdot 1 e^{3-2} = 2e$

$f_z(3,1,2) = -1^2 e^{3-2} = -e$

$D_{\vec{u}} f(3,1,2) = \vec{\nabla} f(3,1,2) \cdot \frac{1}{\sqrt{30}} \langle 2,5,1 \rangle$

$= \langle e, 2e, -e \rangle \cdot \frac{1}{\sqrt{30}} \langle 2,5,1 \rangle = \frac{2e + 10e - e}{\sqrt{30}} = \frac{11e}{\sqrt{30}}$

§14.6 Directional Derivatives and the Gradient Vector (cont'd)

Consider a surface given by $z = f(x,y)$

Q. What is the direction of greatest speed (increase) for $D_{\vec{u}} f(x_0, y_0)$?

In other words, in which direction $\vec{u}$ is $|D_{\vec{u}} f|$ largest and what is its value?

A. $\vec{u} = \frac{\vec{\nabla} f(x_0,y_0)}{|\vec{\nabla} f(x_0,y_0)|}$ is the direction of maximal change

and $|\vec{\nabla} f(x_0,y_0)|$ maximal rate of change.

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Idea: $D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u} = \cos(\theta)\, |\vec{\nabla} f| \cdot |\vec{u}|$

$= \cos(\theta)\, |\vec{\nabla} f| \cdot 1$

where θ angle between $\vec{\nabla} f$ and $\vec{u}$

maximized at $\cos(0) = 1$.

Ex. $f(x,y) = y e^{xy}$. Find direction and magnitude of largest increase at $(0,3)$

Find $\dfrac{\vec{\nabla} f}{|\vec{\nabla} f|}$ and $|\vec{\nabla} f|$ at $(0,3)$

$\vec{\nabla} f(0,3) = \langle f_x(0,3), f_y(0,3)\rangle$

$f_x = y \cdot y e^{xy} = y^2 e^{xy}$

$f_y = e^{xy} + xy e^{xy}$

$f_x(0,3) = 3^2 e^{0\cdot 3} = 9 \cdot 1 = 9$

$f_y(0,3) = e^{0\cdot 3} + 0\cdot 3 \cdot e^{0\cdot 3} = 1 + 0 = 1$

$\vec{\nabla} f(0,3) = \langle 9, 1\rangle$ direction $\dfrac{\langle 9,1\rangle}{\sqrt{9^2+1}} = \dfrac{\langle 9,1\rangle}{\sqrt{82}}$

$|\vec{\nabla} f(0,3)| = \sqrt{82}$

Same works for 3 variables

Ex. $f(x,y,z) = \arctan(xyz)$ at $(1,2,1)$ . . .

Direction and Magnitude of greatest change.

$$\vec{\nabla} f(1,2,1) = \langle f_x(1,2,1), f_y(1,2,1), f_z(1,2,1) \rangle$$

$$f_x = \frac{1}{1+(xyz)^2} \cdot yz \qquad = \frac{1}{1+(2)^2} \cdot 2 \cdot 1 = \frac{2}{5}$$

$$f_y = \frac{1}{1+(xyz)^2} \cdot xz \quad \text{at } (1,2,1) \quad = \frac{1}{1+2^2} \cdot 1 \cdot 1 = \frac{1}{5}$$

$$f_z = \frac{1}{1+(xyz)^2} \cdot xy \qquad = \frac{2}{5}$$

$$\vec{\nabla} f(1,2,1) = \left\langle \frac{2}{5}, \frac{1}{5}, \frac{2}{5} \right\rangle$$

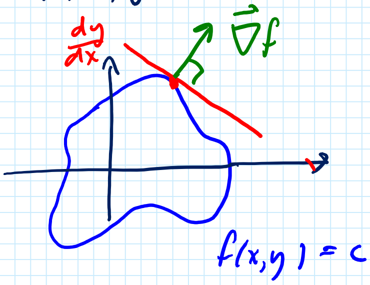
$$|\vec{\nabla} f(1,2,1)| = \sqrt{\left(\frac{2}{5}\right)^2 + \left(\frac{1}{5}\right)^2 + \left(\frac{2}{5}\right)^2} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

$$\vec{u} = \frac{\vec{\nabla} f(1,2,1)}{|\vec{\nabla} f(1,2,1)|} = \frac{5}{3} \left\langle \frac{2}{5}, \frac{1}{5}, \frac{2}{5} \right\rangle = \left\langle \frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right\rangle$$

Answer $|\vec{\nabla} f(1,2,1)| = \frac{3}{5}$, $\vec{u} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \left\langle \frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right\rangle$

Geometrically Understanding the Gradient

$f(x,y) = c$ level curve



Since the change is 0 along any level curve, the max rate of change needs to be $\perp$ to any level curve, that is, $\perp$ to tangent line.

Another way to see this:

slope of tangent $\frac{dy}{dx} = -\frac{f_x}{f_y} = \frac{\text{rise}}{\text{run}}$ so direction of tangent line is $\langle f_y, -f_x \rangle$

$\langle f_y, -f_x \rangle \cdot \vec{\nabla} f$

$\langle f_y, -f_x \rangle \cdot \langle f_x, f_y \rangle = f_x f_y - f_x f_y = 0.$

$k = f(x,y)$ $\quad$ $\vec{r}(t) = \langle x(t), y(t) \rangle$ path on level curve

$\vec{r}\,'(t)$ is in the tangent plane, so $\vec{\nabla} f \cdot \vec{r}\,'(t) = 0$

Same with $k = f(x,y,z)$, $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, so tangent plane is all derivatives of all paths on surface through the point.

Ex. $F(x,y,z) = x^3 + y^3 + z^3 - 6xyz + 2 = 0$ (Implicit Surface)

Find tangent plane at $(2,1,1)$

Any pt (x,y,z) is in tangent plane at $(2,1,1)$ if $\langle x-2, y-1, z-1 \rangle \perp \vec{\nabla} F$

In other words

$$\vec{\nabla} F \cdot \langle x-2, y-1, z-1 \rangle = 0$$

$F_x = 3x^2 - 6yz \quad = 3 \cdot 2^2 - 6 \cdot 1 \cdot 1 = 6$

$F_y = 3y^2 - 6xz \quad$ at $(2,1,1)$ $= 3 \cdot 1^2 - 6 \cdot 2 \cdot 1 = -9$

$F_z = 3z^2 - 6xy \quad = 3 \cdot 1^2 - 6 \cdot 2 \cdot 1 = -9$

so $\langle 6, -9, -9 \rangle \cdot \langle x-2, y-1, z-1 \rangle = 0$ or so $\langle 6, -9, -9 \rangle \cdot \langle x-2, y-1, z-1 \rangle = 0$ or

$$\boxed{6(x-2) - 9(y-1) - 9(z-1) = 0}$$