Math 261, Lecture 16, 9/26/18

- Exam 1 Seating Chart and Study Guide
  available at math.purdue.edu/ma261

Today: §14.7 (begin), Next: §14.7 (finish)

Recap: $z = f(x,y)$ $\quad \vec{\nabla} f = \langle f_x, f_y \rangle$

$$D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u} \qquad |\vec{u}| = 1$$

directional derivative (number)

↳ Direction of fastest increase/decrease at (x_0, y_0)

$$\frac{\vec{\nabla} f(x_0,y_0)}{|\vec{\nabla} f(x_0,y_0)|}$$

↳ Fastest rate of increase/decrease at (x_0,y_0)

$$|\vec{\nabla} f(x_0,y_0)|$$

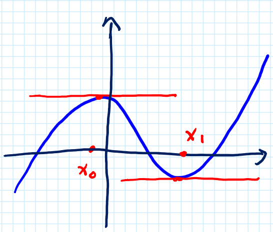
$F(x,y,z) = c$ Implicit Surface

↳ tangent plane at (x_0,y_0,z_0) is

$$F_x(x_0,y_0,z_0)(x - x_0) + F_y(x_0,y_0,z_0)(y - y_0) + F_z(x_0,y_0,z_0)(z - z_0)$$

§14.7 Maximal and Minimal Values

Calc 1 $\quad$ critical pt $\quad$ slope is flat, $f'(x) = 0$



2nd derivative test $\quad f'(a) = 0$

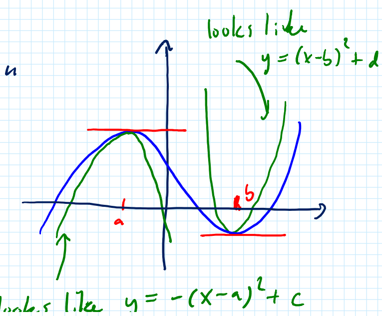
$f''(a) > 0$ local minimum

$f''(a) < 0$ local maximum

$f''(a) = 0$ no information

1st derivative = "best fit line"

2nd derivative = "best fit parabola"



What about 2D? $z = f(x,y)$

local minima and maxima occur where tangent plane is horizontal or "flat"

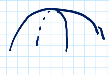
so <u>critical point</u> (a,b) is where $\begin{cases} f_x(a,b) = 0 \\ f_y(a,b) = 0 \end{cases}$

1^{st} derivative = "best fit plane"

2^{nd} derivative = "best fit quadric"

$\hookrightarrow$ $z \approx C(x-a)^2 + D(x-b)^2$

$C, D > 0$  upward elliptic paraboloid so local min.

$C, D < 0$  downward elliptic paraboloid so local max

$C>0, D<0$
$C<0, D>0$  hyperbolic paraboloid "saddle pt" neither local min or local max.

$z = f(x,y)$ (a,b) is a critical pt, $f_x(a,b) = 0 = f_y(a,b)$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx}(a,b) f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$$

"Hessian" at (a,b)

$\hookrightarrow$ Combines all 2^{nd} derivative information in a single <u>number</u>.

$$\boxed{\begin{array}{l} 2^{nd} \text{ Derivative Test} \\ \text{If } (a,b) \text{ a critical pt for } z = f(x,y) \\ \text{Then: } \quad D > 0,\ f_{xx}(a,b) > 0 \quad \text{local min} \\ \qquad\qquad D > 0,\ f_{xx}(a,b) < 0 \quad \text{local max} \\ \qquad\qquad D < 0 \quad \text{saddle pt} \\ \qquad\qquad D = 0 \quad \text{no information} \end{array}}$$

$\ast$ Can use $f_{yy}(a,b)$ to test as well.

Ex. $f(x,y) = -x^3 - 4y^3 + 6xy$.

Find all crit pts, loc. min, loc max, saddle pts.

$$\begin{cases} f_x = -3x^2 + 6y = 0 \rightsquigarrow 6y = 3x^2 \text{ or } y = \frac{1}{2}x^2 \\ f_y = -12y^2 + 6x = 0 \end{cases}$$

sub into 2^{nd} eq'n.

$$-12\left(\tfrac{1}{2}x^2\right)^2 + 6x = 0$$

or $\quad x(x^3 - 2) = 0$

so $\quad x = 0, \sqrt[3]{2}$

So $(0,0)$ and $\left(\sqrt[3]{2}, \frac{\sqrt[3]{4}}{2}\right)$ are the crit pts.

$y = \frac{1}{2}(0)^2 \qquad y = \frac{1}{2}(\sqrt[3]{2})^2$

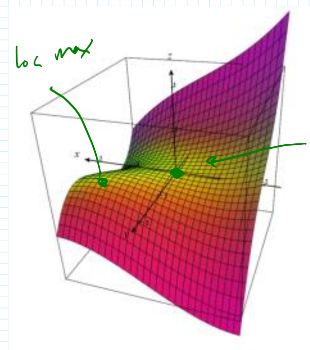
Now find the Hessian:

$f_{xx} = -6x \qquad f_{xy} = f_{yx} = 6$

$f_{yy} = -24y$

$$D = f_{xx} f_{yy} - [f_{xy}]^2 = (-6x)(-24y) - (6)^2 = 144xy - 36$$

$D(0,0) = 144 \cdot 0 \cdot 0 - 36 = -36 < 0$ so $\boxed{(0,0) \text{ saddle pt}}$

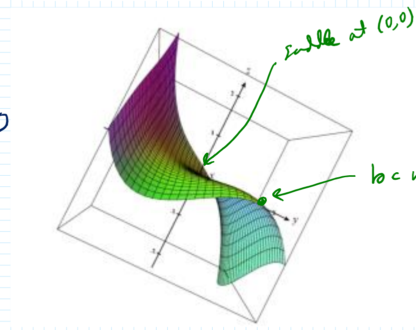


$$D\left(\sqrt[3]{2}, \frac{\sqrt[3]{4}}{2}\right) = 144(\sqrt[3]{2})\left(\frac{\sqrt[3]{4}}{2}\right) - 36$$

$$= 144\left(\frac{\sqrt[3]{8}}{2}\right) - 36 = 144 - 36 = 108 > 0$$

$$f_{xx}\left(\sqrt[3]{2}, \frac{\sqrt[3]{4}}{2}\right) = -6\sqrt[3]{2} < 0$$

so $\boxed{\left(\sqrt[3]{2}, \frac{\sqrt[3]{4}}{2}\right) \text{ is a local max.}}$



two views of a rescaled plot of $z = -x^3 - 4y^3 + 6xy$

Ex. $f(x,y) = x - e^{xy}$. Find crit pts, loc max, loc min, saddle pts.

$\{ f_x = 1 - ye^{xy} = 0 \quad \leftarrow$ sub $x = 0$ into 1^{st} eqn

$$\begin{cases} f_x = 1 - ye^{xy} = 0 \\ f_y = -xe^{xy} = 0 \rightsquigarrow x = 0 \text{ or } \cancel{e^{xy}} = 0 \end{cases}$$

but $e^{(\ )} > 0$

sub $x=0$ into 1st eqn

Subbing $x=0$ into 1st eqn $0 = 1 - ye^0 = 1 - y$ or $y = 1$

So $(0,1)$ only crit pt.

Hessian:

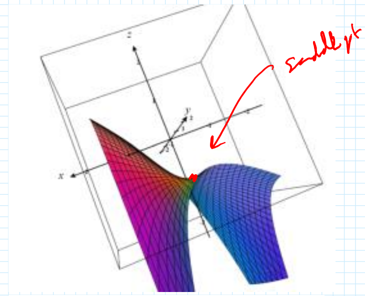
$$f_{xx} = -y^2 e^{xy} \qquad f_{xy} = -e^{xy} - xye^{xy}$$

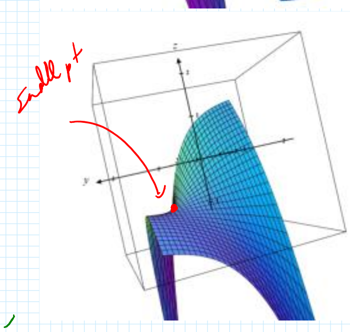
$$f_{yy} = -x^2 e^{xy}$$

$$f_{xx}(0,1) = -(1)^2 e^{0\cdot 1} = -1 \qquad f_{xy}(0,1) = -e^{0\cdot 1} - 0\cdot 1\cdot e^{0\cdot 1} = -1$$

$$f_{yy}(0,1) = -0^2 e^{0\cdot 1} = 0$$

two views of the saddle pt for $z = x - e^{xy}$





$$\text{So } D(0,1) = f_{xx}(0,1) f_{yy}(0,1) - [f_{xy}(0,1)]^2$$

$$= -1\cdot 0 - [-1]^2 = -1$$

So $\boxed{D < 0 \text{ at } (0,1) \text{ which means this is a saddle pt}}$

Bonus Ex. $f(x,y) = y\sin(x)$. Find local min, local max, saddle pts.

$$\begin{cases} f_x = y\cos(x) = 0 \rightsquigarrow y = 0 \text{ or } \boxed{x = \frac{\pi}{2} + \pi k,\ k = 0, 1, -1, 2, -2, \dots} \\ f_y = \sin(x) = 0 \end{cases}$$

$\sin(\frac{\pi}{2} + \pi k) = \pm 1$ so never satisfies 2nd eqn.

So $y = 0$, $\sin(x) = 0$ only solutions or

$y = 0$, $x = \pi \ell$ $\qquad \ell = 0, 1, -1, 2, -2, \dots$

are crit pts.

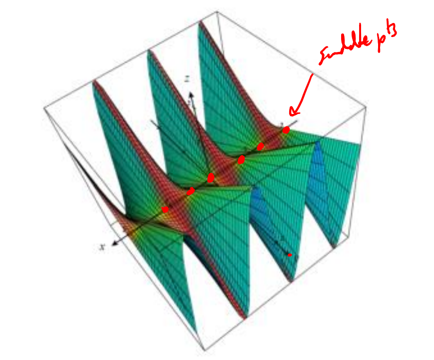
Hessian: $f_{xx} = -y\sin(x) \qquad f_{xy} = \cos(x)$

$$f_{yy} = 0$$

$$D = f_{xx} f_{yy} - [f_{xy}]^2 = -y\sin(x)\cdot 0 - [\cos(x)]^2$$
$$= -\cos(x)^2$$

$$D(\pi \ell, 0) = -\cos(\pi\ell)^2 = -1\,(\pm 1)^2 = -1 < 0$$

so $\boxed{\text{all } (\pi \ell, 0) \text{ are saddle pts}}$



Rescaled graph of

$z = y\sin(x)$

Ex. $f(x,y) = x^2 y e^{-x^2-y^2}$ Find all critical pts.

$$\begin{cases} f_x = 2xy e^{-x^2-y^2} - 2x^3 y e^{-x^2-y^2} = 0 \\ f_y = x^2 e^{-x^2-y^2} - 2x^2y^2 e^{-x^2-y^2} = 0 \end{cases}$$

$e^{-x^2-y^2} > 0$ so factor it out to get

$$\begin{cases} 2xy(1-x^2) = 0 \rightsquigarrow x=0, y=0, x=\pm 1 \\ x^2(1-2y^2) = 0 \end{cases}$$

Try each of these 4 cases in 2nd equation.

Sub $x=0$ get $0^2(1-2y^2)=0$ so any y works!

$(0,y)$ crit pts.

Sub $y=0$ get $x^2(1-2(0)^2) = x^2 = 0$ so
$x=0$

$(0,0)$ is a crit pt (already know from case $x=0$)

Sub $x=1$ get $1^2(1-2y^2)=0$ or $1-2y^2=0$

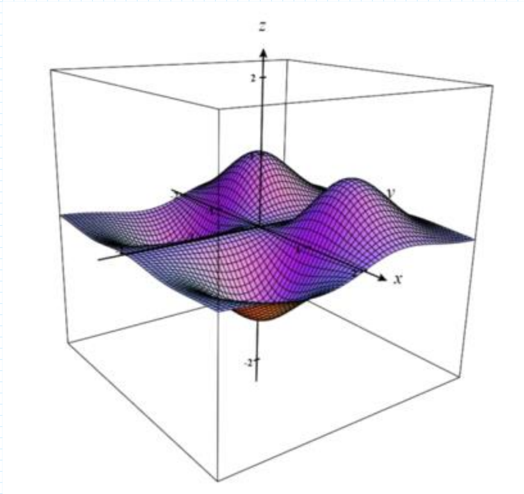
$y=\pm\frac{1}{\sqrt{2}}$

so $\left(1, \pm\frac{1}{\sqrt{2}}\right)$ crit pts

Sub $x=-1$ get $(-1)^2(1-2y^2)=0$ or

$1-2y^2=0$, $y=\pm\frac{1}{\sqrt{2}}$

so $\left(-1, \pm\frac{1}{\sqrt{2}}\right)$ crit pts.



$\left(\pm 1, \frac{1}{\sqrt{2}}\right)$ loc max

$\left(\pm 1, -\frac{1}{\sqrt{2}}\right)$ loc min

$(0,y)$ $y<0$ loc max

$(0,y)$ $y>0$ loc min

$(0,0)$ saddle

rescaled graph of $z=x^2y\,e^{-x^2-y^2}$