Math 261, Lecture 17, 9/28/18

Announcements:

- SI session, T, 5:30 - 6:20 WALC B066
- Office Hours M: 3:30 - 5:30
  T: 1:30 - 3:30
  W: cancelled

Today: §14.7 (finish), Next: Review, Lessons 2-16

Recap: $z = f(x,y)$ a surface

critical pts (a,b) $f_x(a,b) = f_y(a,b) = 0$

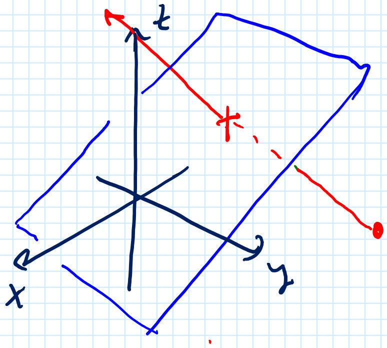
Hessian: $D = f_{xx} f_{yy} - [f_{xy}]^2$

$\star$ 2nd Derivative Test. (a,b) crit pt

- $D > 0$, $f_{xx}(a,b) > 0$ (or $f_{yy}(a,b) > 0$) $\hookrightarrow$ loc. min.
- $D > 0$, $f_{xx}(a,b) < 0$ (or $f_{yy}(a,b) < 0$) $\hookrightarrow$ loc. max.
- $D < 0$ "saddle pt."
- $D = 0$ no info.

§14.7 Min and Max Values (cont'd)

Ex. Find the point on the plane $2x - 3y + z = 0$ of minimal distance to the point $(1, 0, -3)$



distance $(1,0,-3)$ to (x,y,z)

$$d = \sqrt{(x-1)^2 + (y-0)^2 + (z-(-3))^2}$$

↳ solve plane eq'n for z

$z = -2x + 3y$

↳ sub in for z

$$d = \sqrt{(x-1)^2 + y^2 + (-2x+3y+3)^2}$$

Now a function of (x,y) need to minimize

Same as minimizing

$$d^2 = (x-1)^2 + y^2 + (-2x+3y+3)^2$$

Find crit pts

$$(d^2)_x = 2(x-1) + 0 + 2(-2x+3y+3)(-2)$$
$$= 2x - 2 + 8x - 12y - 12$$
$$= 10x - 12y - 14 = 0$$

or $5x - 6y = 7$

$$(d^2)_y = 0 + 2y + 2(-2x+3y+3)\cdot 3$$
$$= 2y - 12x + 18y + 18 = 0$$

$= -12x + 20y + 18 = 0$

or $\quad -6x + 10y = -9$

2 equations in 2 unknowns

$$\begin{cases} 5x - 6y = 7 & \rightsquigarrow \quad x = \frac{7}{5} + \frac{6}{5}y \\ -6x + 10y = -9 & \end{cases}$$

sub into 2nd eq'n

$$-6\left(\frac{7}{5} + \frac{6}{5}y\right) + 10y = -9$$

$$-\frac{42}{5} - \frac{36}{5}y + \frac{50}{5}y = -\frac{45}{5}$$

$$\frac{14}{5}y = -\frac{3}{5} \quad \text{or} \quad y = \frac{-3}{14}$$

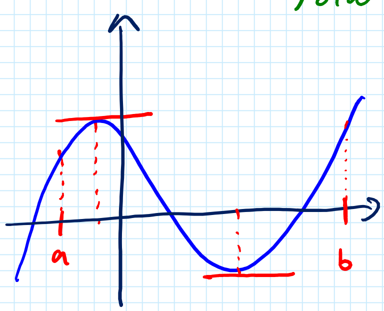
Answer
$$\begin{cases} x = \frac{7}{5} - \frac{6}{5}\left(\frac{-3}{14}\right) = \frac{58}{35} \\ y = -\frac{3}{14} \\ z = -2x + 3y = -\frac{277}{70} \end{cases}$$

This is a minimum since it is the only critical pt and the distance is increasing to ∞ as $x, y,$ or $z \to \infty$.

Absolute Minima and Maxima

Case 1 $\quad y = f(x)$ Find min and max values over a closed interval $[a,b]$

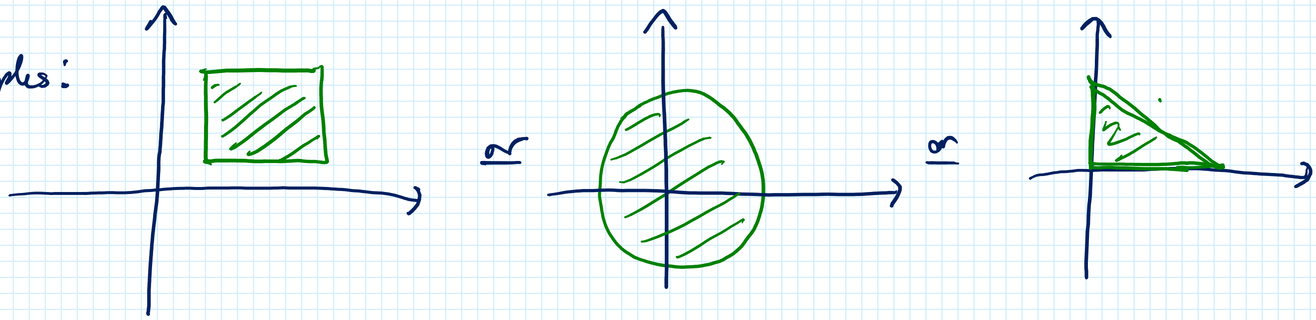
$\hookrightarrow$ Solution is to test all crit pts and end pts.



What happens in the case of two variables?

interval is exchanged for a region in the xy-plane

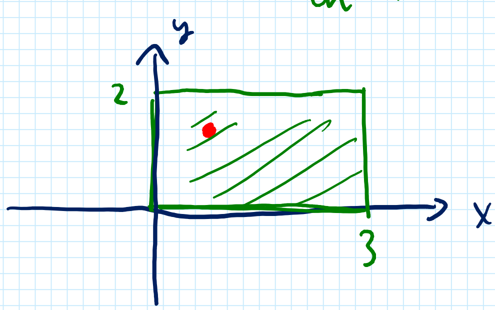
Examples:



✱ If the region has finite extent and contains all points of its boundary, then an absolute max and absolute min are achieved in the region for a function $z = f(x,y)$ ✱

Strategy:
1) Find all crit pts inside region
2) Find all crit pts on the boundary
3) Make sure to check "corner pts"

Ex. Find the absolute max for $f(x,y) = x^2 + y^2 - 2x - 3y + 6$ on the region $0 \le x \le 3$, $0 \le y \le 2$



First find all crit pts.

$$\left.\begin{aligned} f_x &= 2x - 2 = 0 \\ f_y &= 2y - 3 = 0 \end{aligned}\right\} \text{ so }$$

$x = 1$, $y = 3/2$ only crit pt and it belongs to region.

$(1, 3/2)$

There are 4 cases for the boundary:

$x = 0 \quad 0 \le y \le 2$

$f(0,y) = 0^2 + y^2 - 2\cdot 0 - 3y + 6$

$= y^2 - 3y + 6$ find crit pts $0 \le y \le 2$

$f'(0,y) = 2y - 3 = 0 \quad y = 3/2$, plus endpts $y = 0, 2$

or $(0,0), (0,3/2), (0,2)$

$x = 2, \quad 0 \le y \le 2$

$f(2,y) = 2^2 + y^2 - 2\cdot 2 - 3y + 6$

$= y^2 - 3y + 6$

$f'(2,y) = 2y - 3 = 0$ so crit pts are $y = 0, 3/2, 2$

so $(2,0), (2,3/2), (2,2)$

$y = 0 \quad 0 \le x \le 3$

$f(x,0) = x^2 - 2x + 6$

$f'(x,0) = 2x - 2 = 0$ so $x = 1$ and endpts $x = 0, 3$

so test $(0,0), (1,0), (3,0)$

$y = 2 \quad 0 \le x \le 3$

$f(x,2) = x^2 + (2)^2 - 2x - 3\cdot 2 + 6$

$= x^2 - 2x + 4$

$f'(x,2) = 2x - 2 = 0$ or $x = 1$ and endpts $x = 0, 3$

so test $(0,2), (1,2), (3,2)$.

All together, we have the following test pts:

crit pt $\longrightarrow$ $(1, 3/2)$

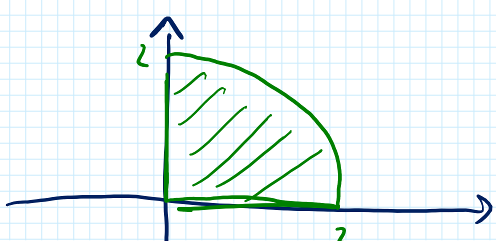
"corners" $\left\{ \begin{array}{l} (0,0) \\ (0,2) \\ (3,0) \\ (3,2) \end{array} \right.$

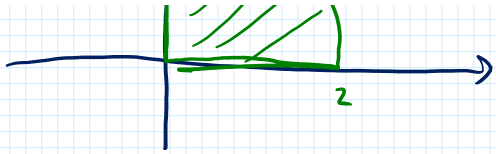
$\left. \begin{array}{l} (0, 3/2) \\ (3, 3/2) \\ (1,0) \\ (1,2) \end{array} \right\}$ on the edges of the rectangle

$$f(1, 3/2) = 1 + \left(\tfrac{3}{2}\right)^2 - 2\cdot 1 - 3\left(\tfrac{3}{2}\right) + 6 = 11/4$$
$$f(0,0) = 0^2 + 0^2 - 2\cdot 0 - 3\cdot 0 + 6 = 6$$
$$f(0,2) = 0^2 + (2)^2 - 2\cdot 0 - 3\cdot 2 + 6 = 4$$
$$f(3,0) = 3^2 + 0^2 - 2\cdot 3 - 3\cdot 0 + 6 = 9$$
$$f(3,2) = 3^2 + 2^2 - 2\cdot 3 - 3\cdot 2 + 6 = 7$$
$$f(0, 3/2) = 0^2 + \left(\tfrac{3}{2}\right)^2 - 2\cdot 0 - 3\left(\tfrac{3}{2}\right) + 6 = 15/4$$
$$f(3, 3/2) = 3^2 + \left(\tfrac{3}{2}\right)^2 - 2\cdot 3 - 3\cdot\tfrac{3}{2} + 6 = 25/4$$
$$f(1,0) = 1^2 + 0^2 - 2\cdot 1 - 3\cdot 0 + 6 = 5$$
$$f(1,2) = 1^2 + 2^2 - 2\cdot 1 - 3\cdot 2 + 6 = 4$$

max value at $(3,0)$

Bonus Ex. Find Max and Min Values for $f(x,y) = x^2 y$ on region $x, y \geq 0$, $\sqrt{x^2+y^2} \leq 2$





$$\left.\begin{aligned} f_x &= 2xy = 0 \\ f_y &= x^2 = 0 \end{aligned}\right\}$$ so $x=0, y=0$ in region.

3 cases for boundary.

$y=0 \quad 0 \le x \le 2$

$f(x,0) = 0$

$f'(x,0) = 0$ so $(x,0)$, $0 \le x \le 2$, test pts.

$x=0,\ 0 \le y \le 2$

$f(0,y) = 0$, $f'(0,y) = 0$, so all $(0,y)$, $0 \le y \le 2$, test pts.

$\sqrt{x^2+y^2} = 2$, $0 \le y \le 2$, $0 \le x \le 2$ quarter circle

same as $x^2+y^2=4$,

or $x^2 = 4-y^2$, $0 \le y \le 2$

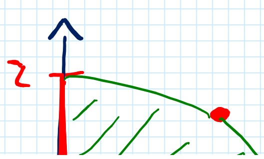
$f(x,y) = x^2y$ or $(4-y^2)y$ on quarter circle

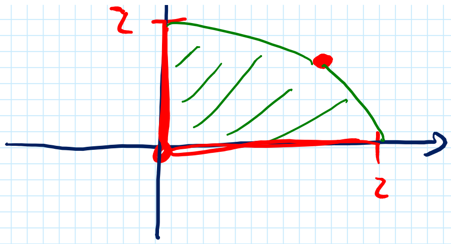
$f(x,y) = 4y - y^3$ on quarter circle of

$f'(y) = 4 - 3y^2$ or $y = \pm \frac{2}{\sqrt{3}}$; $0 \le y \le 2$

so $\left(\sqrt{4 - \left(\frac{2}{\sqrt{3}}\right)^2}, \frac{2}{\sqrt{3}}\right)$ test pt.

or $\left(\frac{2\sqrt{2}}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$





test pts in red.

On the x- or y-axis $f(x,y) = x^2 y$ is $\boxed{0}$ is min

$$f\left(\frac{2\sqrt{2}}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}\right) = \frac{8}{3}\frac{\sqrt{2}}{\sqrt{3}} = \boxed{\frac{8\sqrt{2}}{3\sqrt{3}}} \text{ is max}$$