Math 261, Lecture 19, 10/3/18

- NO CLASS 10/5, 10/8, 10/9

Today: §14.8 (begin)

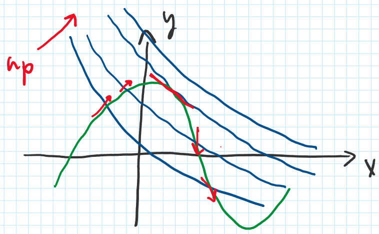
§14.8 Langrange Multipliers

↳ Absolute Maxima / Minima

$z = f(x,y)$ surface

maximize / minimize on a region

Method of Finding Abs Max / Min on a constraint curve.



$z = f(x,y)$ surface

$c = g(x,y)$

↳ constraint curve

$\boxed{\text{tangent to curve parallel to tangent to level curve } k = f(x,y) \text{ for } k \text{ to be a max/min value}}$

How do we implement?

tangents in "same direction" is the same as gradients in "same direction"

$$\boxed{\vec{\nabla} f = \lambda \vec{\nabla} g}$$

baby lambda, Λ big Lambda

Goal. Solve this subject to $c = g(x,y)$.

Ex. $f(x,y) = x^2 y$ Max / Min with respect to constraint $2x^2 + y^2 = 3$

$3 = g(x,y) = 2x^2+y^2$

$$\vec{\nabla} f = \lambda \vec{\nabla} g$$

$$\vec{\nabla} f = \langle f_x, f_y \rangle = \langle 2xy, x^2 \rangle$$

$$\vec{\nabla} g = \langle g_x, g_y \rangle = \langle 4x, 2y \rangle$$

LM $\begin{cases} 2xy = 4\lambda x \\ x^2 = 2\lambda y \\ 2x^2+y^2 = 3 \end{cases}$

$2xy = 4\lambda x \longrightarrow x = 0$, $x \neq 0$: $2\cancel{x}y = 4\lambda \cancel{x}$, $y = 2\lambda$

$x^2 = 2\lambda \cdot 2\lambda = 4\lambda^2$

$x = \pm 2\lambda$

$(\pm 2\lambda, 2\lambda)$

$2(\pm 2\lambda)^2 + (2\lambda)^2 = 3$

$8\lambda^2 + 4\lambda^2 = 3$

$12\lambda^2 = 3$, $\lambda^2 = \frac{1}{4}$

$\lambda = \pm \frac{1}{2}$

$x = 0$, $2(0)^2 + y^2 = 3$

$y = \pm\sqrt{3}$

$(0, \pm\sqrt{3})$

$(\pm 1, 1)$ $\lambda = \frac{1}{2}$

$(\pm 1, -1)$ $\lambda = -\frac{1}{2}$

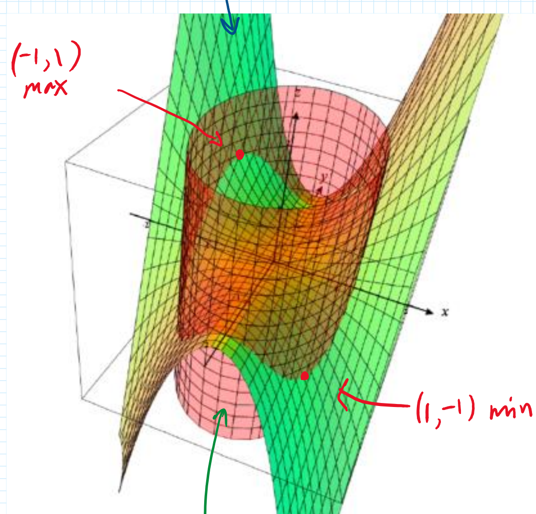
$f(x,y) = x^2 y$

$(0, \pm\sqrt{3}) = 0^2 \cdot \pm\sqrt{3} = 0$

$(\pm 1, 1) = (\pm 1)^2 \cdot 1 = 1$ ← Max

$(\pm 1, -1) = (\pm 1)^2 (-1) = -1$ ← Min

$f(x,y) = x^2 y$



$(-1,1)$ max

$(1,-1)$ min

constraint $2x^2+y^2=3$

Ex. $x^2+y^2+3z^2 = 6$ ellipsoid. Find the pts of max/min distance on the ellipsoid to $(-1,1,0)$

$f(x,y,z) = d^2(x,y,z) = (x+1)^2 + (y-1)^2 + (z-0)^2$

$g(x,y,z) = x^2+y^2+3z^2 = 6$ constraint.

$$\vec{\nabla} f = \lambda \vec{\nabla} g$$

LM $\begin{cases} 2(x+1) = 2\lambda x \\ 2(y-1) = 2\lambda y \\ 2z = 6\lambda z \\ x^2+y^2+3z^2 = 6 \end{cases}$

$2z = 6\lambda z \longrightarrow z = 0$, $z \neq 0$: $\lambda = 1/3$

$\lambda = \frac{1}{3}$ $\quad 2(x+1) = \frac{2}{3}x \rightsquigarrow \frac{4}{3}x + 2 = 0$

$x^2 + y^2 + 3z^2 = 6$

$\lambda = \frac{1}{3}$ $\quad 2(x+1) = \frac{2}{3}x \rightsquigarrow \frac{4}{3}x + 2 = 0$

$\rightsquigarrow x = -3/2$

$2(y-1) = \frac{2}{3}y \rightsquigarrow y = 3/2$

$$\left(\frac{-3}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + 3z^2 = 6$$

$$\frac{9}{2} + 3z^2 = 6$$

$$3z^2 = \frac{3}{2} \quad , z = \frac{\pm 1}{\sqrt{2}}$$

$\left(-\frac{3}{2}, \frac{3}{2}, \pm\frac{1}{\sqrt{2}}\right)$ are test pts

$z = 0$
$$\begin{cases} 2(x+1) = 2\lambda x \rightsquigarrow x+1 = \lambda x \text{ or } x = \frac{-1}{1-\lambda} \\ 2(y-1) = 2\lambda y \rightsquigarrow y-1 = \lambda y \text{ or } y = \frac{1}{1-\lambda} \\ x^2 + y^2 = 6 \end{cases}$$

Sub in to here

$$\left(\frac{-1}{1-\lambda}\right)^2 + \left(\frac{1}{1-\lambda}\right)^2 = 6$$

$$\frac{1}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} = 6$$

$$\frac{1}{(1-\lambda)^2} = 3 \text{ or } (1-\lambda)^2 = \frac{1}{3}$$

$$\lambda = 1 \pm \frac{1}{\sqrt{3}}$$

so $\quad x = \frac{-1}{1-\lambda} = \mp\sqrt{3}$

$y = \frac{1}{1-\lambda} = \pm\sqrt{3}$

$(-\sqrt{3}, \sqrt{3}, 0)$ and $(\sqrt{3}, -\sqrt{3}, 0)$ are test pts

$$f(x,y,z) = (x+1)^2 + (y-1)^2 + z^2$$

$$f\left(-\frac{3}{2}, \frac{3}{2}, \pm\frac{1}{\sqrt{2}}\right) = \left(\frac{-1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\pm\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{4} + \frac{1}{4} + \frac{1}{2} = 1 \quad \text{min}$$

$$f(-\sqrt{3}, \sqrt{3}, 0) = (1-\sqrt{3})^2 + (\sqrt{3}-1)^2 + 0^2 = 2(2\sqrt{3}-2)$$
$$\approx 2.92820$$

$$f(\sqrt{3}, -\sqrt{3}, 0) = (\sqrt{3}+1)^2 + (-\sqrt{3}-1)^2 + 0^2 = 2(4+2\sqrt{3})$$
$$\approx 14.92820 \quad \text{max}$$