Math 261, Lecture 21, 10/12/18 Today §15.2 (all), Next: §15.3 (all)

Recap $z = f(x,y)$ $R = \left\{ (x,y) : \begin{matrix} a \le x \le b \\ c \le y \le d \end{matrix} \right.$ Rectangle in xy-plane

"Volume"

$$\iint_R f(x,y)\, dA = \int_c^d \int_a^b f(x,y)\, dx\, dy = \int_a^b \int_c^d f(x,y)\, dy\, dx$$

area of small rectangle

dx AD wrt $\frac{\partial}{\partial x}$

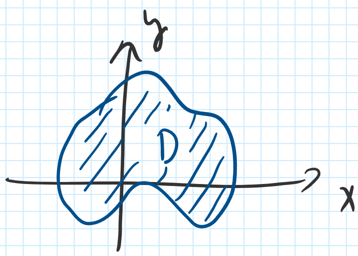
dy AD wrt $\frac{\partial}{\partial y}$

$\iint_R 1 \, dA$?

box height $= 1$, base R

Volume $= 1 \times$ Area of R $=$ Area of R.

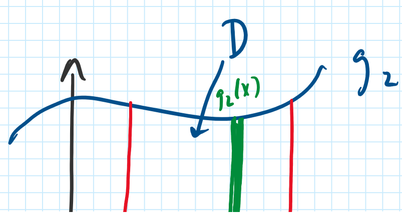
§15.2 Double Integrals over General Regions



$$\iint_D f(x,y)\, dA$$

Two special types

Type I

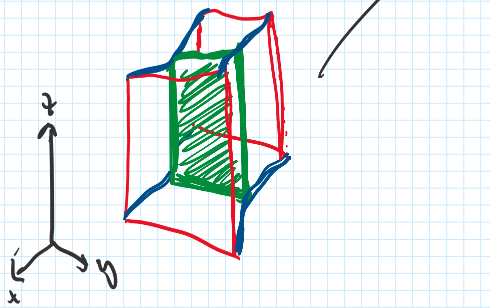


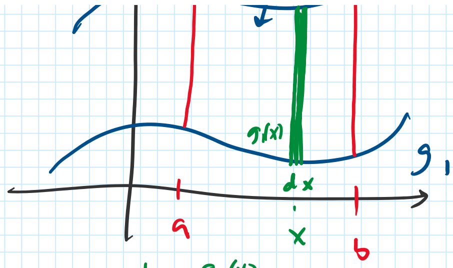
$a \le x \le b$

$g_1(x) \le y \le g_2(x)$

$g_1(x) \le y \le g_2(x)$

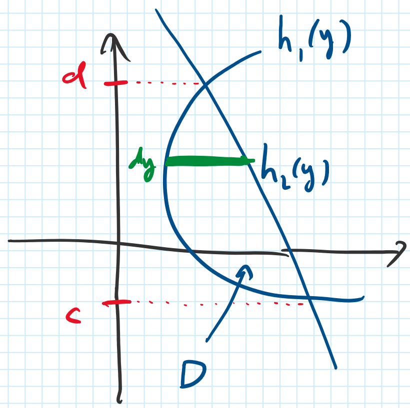
Type I





$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

Type II



$h_1(y) \le x \le h_2(y)$

$c \le y \le d$

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$$

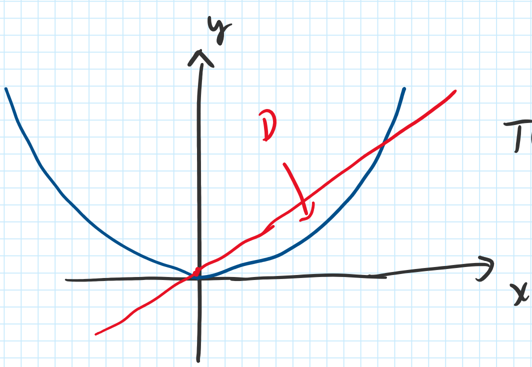
numbers are last

* Not so easy to swap limits!

Ex. $f(x,y) = x^2 y + 2xy$ Integrate over region D

defined by $y = \frac{1}{2}x^2$

$y = 2x$



Type I

Find extent on x-axis

curves intersect at $(0,0)$
$(4,8)$

$\frac{1}{2}x^2 = y = 2x$

$x^2 - 4x = 0$

$\frac{1}{2}x^2 = y = 2x$

$x^2 = 4x \rightsquigarrow x^2 - 4x = 0$

$x(x-4)$

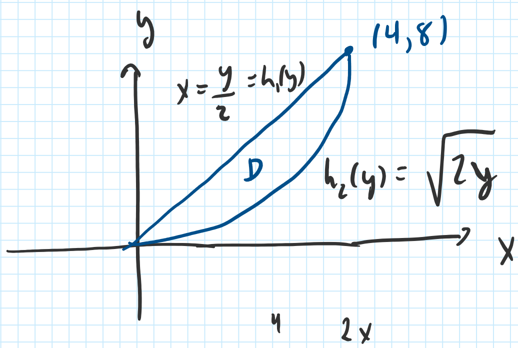
$$\int_0^4 \int_{y=\frac{1}{2}x^2}^{2x} (x^2y + 2xy)\,dy\,dx$$

$$= \int_0^4 \left[\frac{1}{2}x^2y^2 + xy^2 \Big|_{y=\frac{1}{2}x^2}^{2x} \right] dx$$

$$\int_0^4 \left[\frac{1}{2}x^2(2x)^2 + x(2x)^2 - \frac{1}{2}x^2\left(\frac{1}{2}x^2\right)^2 - x\left(\frac{1}{2}x^2\right)^2 \right] dx$$

$$= \int_0^4 2x^4 + 4x^3 - \frac{1}{8}x^6 - \frac{1}{4}x^5 \; dx$$

Ex.



Also Type II

$$\int_0^4 \int_{y=\frac{1}{2}x^2}^{2x} (x^2y + 2xy)\,dy\,dx$$

$$= \int_0^8 \int_{x=\frac{y}{2}}^{\sqrt{2y}} (x^2y + 2xy)\,dx\,dy$$

$$0 \quad x=\tfrac{y}{2}$$

$$= \int_0^8 \left[\tfrac{1}{3}x^3 y + x^2 y\right]_{x=\frac{y}{2}}^{\sqrt{2y}} dy = \int_0^8 \frac{4\sqrt{2}}{3} y^{5/2} + 2y^3 - \frac{1}{24}y^4 - \frac{1}{4}y^3 \, dy$$

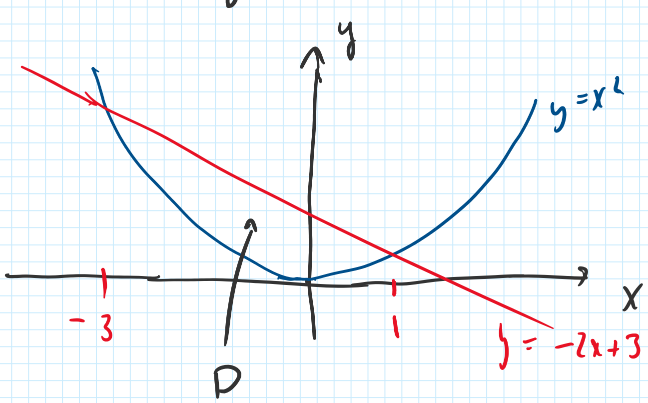
Ex. Find the volume of the region bounded by

plane $\quad 4x + 2y + z - 6 = 0$

cylinder $\quad y = x^2$

xy plane $\quad z = 0$

Find region on the xy-axis bounded by plane and cylinder



Type I

not Type II

* In plane $z = 6$ at $(x,y) = (0,0)$
So plane is above xy plane in the intersection

* Set $z = 0$ in plane eq'n to find intersection with xy-plane

$$4x + 2y - 6 = 0$$

$$2y = -4x + 6$$

$$y = -2x + 3$$

$$x^2 = -2x + 3 \rightarrow x^2 + 2x - 3 = 0$$

$$(x-1)(x+3) = 0 \rightarrow x = 1, -3$$

$$\int_{-3}^{1} \int_{y=x^2}^{-2x+3} (6 - 4x - 2y) \, dy \, dx$$

plane $\quad 4x + 2y + z - 6 = 0$

$$z = 6 - 4x - 2y$$

1

plane $4x + 2y + z - 6 = 0$

$$z = 6 - 4x - 2y$$

$$= \int_{-3}^{1} \left[6y - 4xy - y^2 \Big|_{y=x^2}^{-2x+3} \right] dx$$

$$\int_{-3}^{1} \left(6(-2x+3) - 4x(-2x+3) - (-2x+3)^2 - 6x^2 + 4x^3 + x^4\right) dx$$

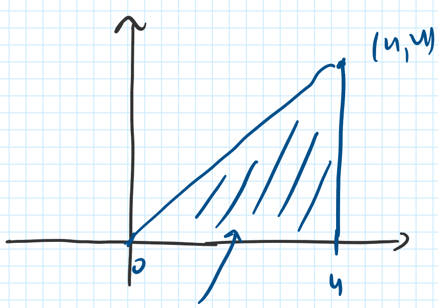
$$= \int_{-3}^{1} \left(-\cancel{12x} + 18 + 8x - 12 - 4x^2 + \cancel{12x} - 9 - 6x^2 + 4x^3 + x^4\right) dx$$

$$= \int_{-3}^{1} x^4 + 4x^3 - 10x^2 + 8x - 3 \, dx$$

$$A(D) = \iint_D 1 \, dA \quad \text{area of region}$$

<u>Average value</u> f over D $\quad \frac{1}{A(D)} \iint_D f(x,y)\, dA$

Ex. $f(x,y) = x^2 e^{xy}$ Avg value over $0 \le x \le 4$, $0 \le y \le x$



$A(D) = 8$

$$\frac{1}{8} \int_0^4 \int_0^x x^2 e^{xy} \, dy \, dx$$

As Type I $\rightarrow$

$$= \frac{1}{8} \int_0^4 \left[x e^{xy} \right]_{y=0}^{x} dx$$

$$= \frac{1}{8} \int^4 x e^{x^2} - x \quad dx$$

$$= \frac{1}{8} \int_0^4 x e^{x^2} - x \; dx$$

$$= \frac{1}{8} \left[\frac{1}{2} e^{x^2} - \frac{1}{2} x^2 \right] \Big|_0^4$$

As Type II $\displaystyle \int_0^4 \int_{x=y}^4 x^2 e^{xy} \, dx \, dy$

This is doable, but requires integration by parts twice!