Math 261, Lecture 23, 10/17/18

Recap. Polar coordinates

$r^2 = x^2+y^2$, $x = r\cos\theta$, $y = r\sin\theta$

$$\iint_D f(x,y)\,dA = \iint_D f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

↑ don't forget!

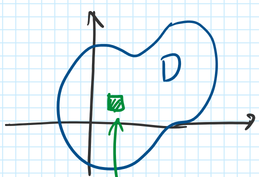
$$* \quad \cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

$$* \quad \sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

Today: §§15.4, 15.5, Next: §15.6

§15.4, Applications of Double Integrals

Center of Mass Problems



"thin plate" or lamina

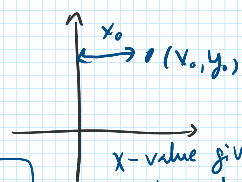
density = mass/area

$\rho(x,y)$

ρ = baby rho

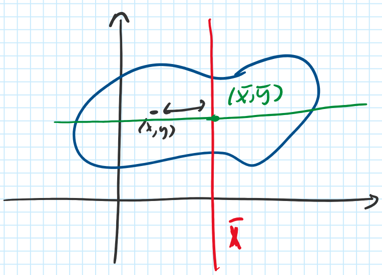
mass of small square $\rho(x,y)\,dA$

$$m = \text{Mass of Lamina} = \iint_D \rho(x,y)\,dA$$



x-value gives distance to y-axis

$$\boxed{\begin{array}{ll} \text{Moment about } y\text{-axis} & M_y = \iint_D x\,\rho(x,y)\,dA \\ \text{Moment about } x\text{-axis} & M_x = \iint_D y\,\rho(x,y)\,dA \end{array}}$$



$$\iint_D (x-\bar{x})\,\rho(x,y)\,dA = 0$$

balanced on fulcrum

$$\iint_D x\,\rho(x,y)\,dA = \bar{x}\iint_D \rho(x,y)\,dA$$

$\quad\quad\quad\quad \| \quad\quad\quad\quad\quad\quad \| \; m$

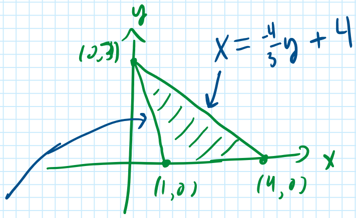
$$\bar{x} = \frac{M_y}{}$$

$$\iint_D x\rho(x,y)\,dA = \bar{x}\iint_D \rho(x,y)\,dA$$

$$\underbrace{\iint_D x\rho(x,y)\,dA}_{M_y} \qquad \underbrace{\iint_D \rho(x,y)\,dA}_{m}$$

$$\boxed{\begin{array}{l} \bar{x} = \dfrac{M_y}{m} \\[2ex] \bar{y} = \dfrac{M_x}{m} \\[2ex] (\bar{x},\bar{y}) \text{ center of mass} \end{array}}$$

Ex.



Find center of mass

density proportional to distance from y-axis.

$x = -\frac{1}{3}y + 1$ Type II Region

$\rho(x,y) = kx$ since ρ is proportional to distance to y-axis and we are in the first quadrant

* For simplicity, let $k=1$ so $\rho(x,y) = x$

$$\bar{x} = \frac{M_y}{m}, \quad \bar{y} = \frac{M_x}{m}$$

$$m = \int_{y=0}^{3}\int_{x=-\frac{1}{3}y+1}^{-\frac{4}{3}y+4} x \, dx\, dy$$

$$= \int_{y=0}^{3}\left[\frac{1}{2}x^2\Big|_{x=-\frac{1}{3}y+1}^{-\frac{4}{3}y+4}\right] dy$$

$$= \int_0^3 \frac{1}{2}\left(-\frac{4}{3}y+4\right)^2 - \frac{1}{2}\left(-\frac{1}{3}y+1\right)^2 \, dy$$

$$= \frac{1}{2}\left[-\frac{1}{4}\left(-\frac{4}{3}y+4\right)^3 + \left(-\frac{1}{3}y+1\right)^3\right]\Big|_{y=0}^{3} = \frac{1}{2}\left[-0+0+\frac{1}{4}\left(\frac{8}{3}\right)^3 - \left(\frac{2}{3}\right)^3\right]$$

$$= \frac{64}{27} - \frac{8}{27} = \frac{56}{27}$$

$$M_y = \iint_D x\rho(x,y)\,dA = \int_{y=0}^{3}\int_{x=\frac{1}{3}y+1}^{-\frac{4}{3}y+4} x^2\,dx$$

$$= \frac{1}{3}\int_{y=0}^{3}\left(-\frac{4}{3}y+4\right)^3 - \left(-\frac{1}{3}y+1\right)^3 dy = -\frac{1}{16}\left(-\frac{4}{3}y+4\right)^4 + \frac{1}{4}\left(-\frac{1}{3}y+1\right)^4\Big|_{y=0}^{3}$$

$$= -0+0+\frac{1}{16}\left(\frac{8}{3}\right)^4 - \frac{1}{4}\left(\frac{2}{3}\right)^4 = \frac{256}{81} - \frac{4}{81} = \frac{252}{81}$$

$$M_x = \iint y\rho(x,y)\,dA = \int_{y=0}^{3}\int_{x=-\frac{1}{3}y+1}^{-\frac{4}{3}y+4} xy\,dx\,dy$$

$$M_x = \iint_D y\,\rho(x,y)\,dA = \int_{y=0} \int_{x=-\frac{1}{3}y+1} xy \; dx\,dy$$

$$= \frac{1}{2}\int_{y=0}^{3} y\left[\left(-\frac{4}{3}y+4\right)^2 - \left(-\frac{1}{3}y+1\right)^2\right] dy = \frac{1}{2}\int_0^3 \frac{15}{9}y^3 - 10y^2 + 15y\,dy$$

$$= \frac{1}{3}\left[\frac{15}{36}y^4 - \frac{10}{3}y^3 + \frac{15}{2}y^2\right]_{y=0}^{3}$$

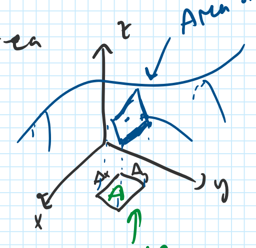
$$= \frac{1}{3}\left[\frac{135}{4} - 90 + \frac{135}{2} - 0 - 0 - 0\right]$$

$$= \frac{15}{4}$$

$$\bar{x} = \frac{M_y}{m} = \frac{252}{81}\cdot\frac{27}{56} = 1.5$$

$$\bar{y} = \frac{M_x}{m} = \frac{15}{4}\cdot\frac{27}{56} \approx 1.808040$$

§15.5 Surface Area



$$SA = \iint_D \left(\frac{dR}{dA}\right) dA$$

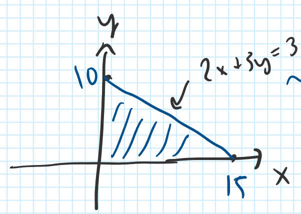
Ratio of areas of squares

Magically, it works out to this formula:

$$SA = \iint_D \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\; dA$$

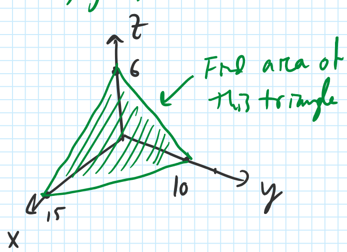
If $F(x,y,z) = z - f(x,y) = 0$ is the implicit surface equation, then $SA = \iint_D |\vec{\nabla} F|\,dA$!

Ex. Surface area of triangle given by plane $2x + 3y + 5z = 30$ in $x \geq 0, y \geq 0, z \geq 0$



$2x + 3y = 30 \rightsquigarrow y = -\frac{2}{3}x + 10$

Base Region



$$z = 6 - \frac{2}{5}x - \frac{3}{5}y$$

$$\frac{\partial z}{\partial x} = -\frac{2}{5}, \quad \frac{\partial z}{\partial y} = -\frac{3}{5}$$

$$SA = \int_{x=0}^{15} \int_{y=0}^{-\frac{2}{3}x+10} \sqrt{\left(-\frac{2}{5}\right)^2 + \left(-\frac{3}{5}\right)^2 + 1}\; dy\,dx$$

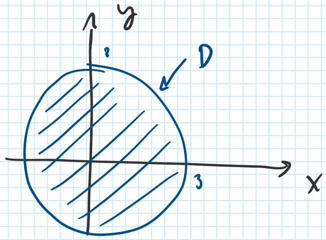
$$= \int_{x=0}^{15} \int_{y=0}^{-\frac{2}{3}x+10} \sqrt{\frac{38}{25}}\; dy\,dx$$

$$= \int_{x=0} \int_{y=0} \sqrt{\frac{38}{25}}\, dy\, dx$$

$$= \frac{\sqrt{38}}{5} \int_{x=0}^{15} \left(-\frac{2}{3}x + 10 - 0\right) dx$$

$$= \frac{\sqrt{38}}{5} \left[-\frac{1}{3}x^2 + 10x\right]\Bigg|_{x=0}^{15} = \frac{\sqrt{38}}{5}\left(150 - \frac{1}{3}(15)^2\right) = \sqrt{38}\cdot 15$$

Ex. surface area of $z = x^2 - y^2$ over $x^2 + y^2 \le 9$



$$\frac{\partial z}{\partial x} = 2x \ , \ \frac{\partial z}{\partial y} = -2y$$

$$SA = \iint_D \sqrt{(2x)^2 + (-2y)^2 + 1}\ dA$$

$$= \iint_D \sqrt{1 + 4x^2 + 4y^2}\ dA$$

Convert to polar $x^2 + y^2 \le 9 \rightsquigarrow r^2 \le 9$ or $0 \le r \le 3$
since $r \ge 0$ always

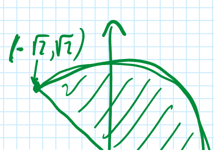
$0 \le \theta \le 2\pi$ since region D extends in all directions

$$SA = \int_{\theta=0}^{2\pi}\int_{r=0}^{3} r\sqrt{1+4r^2}\ dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[\frac{1}{8}\cdot\frac{2}{3}\left(1+4r^2\right)^{3/2}\right]_{r=0}^{3} d\theta$$

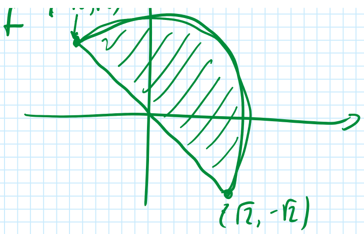
$$= \frac{1}{12}\left[(37)^{3/2} - 1\right] \int_0^{2\pi} d\theta = \frac{\pi}{6}\left[(37)^{3/2} - 1\right]$$

Bonus Ex. Find center of mass of the semicircle



Bonus Ex. Find center of mass of the semicircle

where density is proportional to distance from origin.



$\rho(x,y) = k\sqrt{x^2+y^2}$

$(\sqrt{2}, -\sqrt{2})$ on semicircle, so radius $= \sqrt{(\sqrt{2})^2 + (-\sqrt{2})^2} = \sqrt{2+2} = \sqrt{4} = 2$

Since $r = 2$ we have $(\sqrt{2}, -\sqrt{2}) = (2\cos\theta, 2\sin\theta)$

or $\theta = -\pi/4$

For $(-\sqrt{2}, \sqrt{2}) = (2\cos\theta, 2\sin\theta)$ or $\theta = \frac{3\pi}{4}$

So $-\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$

Since region is a semicircle of radius 2 at origin

$0 \le r \le 2$ always

$$m = \iint_D \rho(x,y)\, dA = \int_{\theta=-\pi/4}^{3\pi/4}\int_{r=0}^{2} \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2} \cdot r\, dr\, d\theta$$

$$\boxed{\begin{gathered}\rho(x,y) = \sqrt{x^2+y^2} \\ \text{or } \rho(r,\theta) = r\end{gathered}}$$

$$= \int_{-\pi/4}^{3\pi/4}\int_0^2 r^2\, dr\, d\theta$$

$$= \left[\frac{1}{3}r^3\Big|_{r=0}^{2}\right] \cdot \int_{-\pi/4}^{3\pi/4} d\theta = \frac{8\pi}{3}$$

$$M_y = \iint_D x\,\rho(x,y)\, dA = \int_{-\pi/4}^{3\pi/4}\int_0^2 r\cos\theta \cdot r \cdot r\, dr\, d\theta$$

$$= \frac{1}{4}r^4\Big|_{r=0}^{2} \int_{-\pi/4}^{3\pi/4} \cos\theta\, d\theta$$

$$= 4\sin\theta\Big|_{\theta=-\pi/4}^{3\pi/4} = 4\left[\frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right)\right] = 4\sqrt{2}$$

$$M_x = \iint_D y\,\rho(x,y)\, dA = \int_{-\pi/4}^{3\pi/4}\int_0^2 r\sin\theta \cdot r \cdot r\, dr\, d\theta$$

$$= \left[\frac{1}{4}r^4\Big|^2\right] \int^{3\pi/4} \sin\theta\, d\theta$$

$$= \left[\frac{1}{4}r^4\Big|_{r=0}^{2}\right]\int_{-\pi/4}^{3\pi/4} \sin\theta \, d\theta$$

$$= -4\cos\theta\Big|_{\theta=-\pi/4}^{3\pi/4} = 4\sqrt{2}$$

$$\bar{x} = \frac{M_y}{m} = 4\sqrt{2}\cdot\frac{3}{8\pi} = \frac{3\sqrt{2}}{2\pi}$$

$$\bar{y} = \frac{M_x}{m} = 4\sqrt{2}\cdot\frac{3}{8\pi} = \frac{3\sqrt{2}}{2\pi}$$