Math 261, lecture 28, 10/29/18

- EXAM 2 is WED, 6:30PM ELLIOTT
  - Show up 10-15 minutes early to get seated and fill out scantron

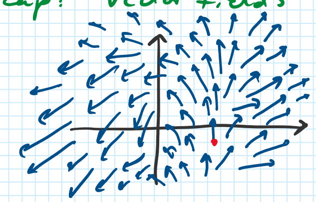
- Office Hours (in MATH 744)
  - Today (10/29) 3:30-5:30
  - Tues (10/30) 2:30-4:30
  - Wed (10/31) 3:30-5:30

* Lagrange Multipliers!

* 14.7

Today: § 16.2 (begin), Next: Exam 2 Review

Recap: Vector fields $F(x,y)$ is a 2D vector or $F(x,y,z)$ a 3D vector



$w(x,y)$ wind

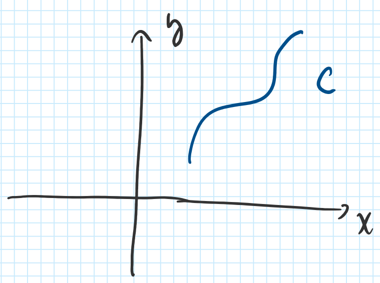
$|w(x,y)|$ wind <u>speed</u> at (x,y)

$\dfrac{w(x,y)}{|w(x,y)|}$ wind <u>direction</u> at (x,y)

$$F(x,y) = \underbrace{\vec{\nabla} f(x,y)}_{\text{potential for } F} = \langle f_x, f_y \rangle$$

F is <u>conservative</u> if it has a potential

§ 16.2 Line Integrals

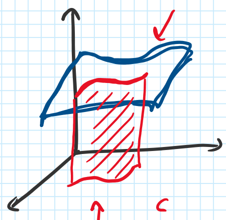


C, a curve, is parametrized by $(x(t), y(t))$, $a \le t \le b$

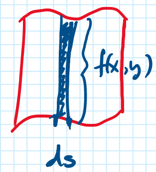
$z = f(x,y)$ surface above xy plane

↑ ✓ $(x(t), y(t), f(x(t), y(t)))$

$z = f(x,y)$ surface above xy plane



line integral is area



$$= \int_C f(x,y)\, ds$$

ds is the differential on arc length

$$s = \text{arc length } C = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

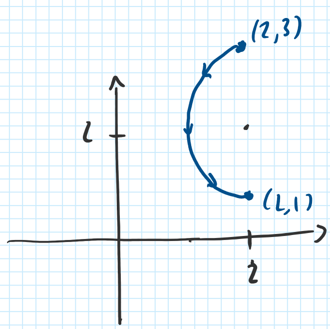
$$\boxed{\int_C f(x,y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

Line integral over C wrt arc length

* Will not depend on choice of parametrization.

Ex. $C \begin{cases} x(t) = 2 + \cos(t) \\ y(t) = 2 + \sin(t) \end{cases} \quad \frac{\pi}{2} \le t \le \frac{3\pi}{2}$

$$\int_C xy\, ds$$



$\frac{\pi}{2} \le t \le \frac{3\pi}{2}$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad \frac{dx}{dt} = -\sin(t) \qquad \frac{dy}{dt} = \cos(t)$$

$$ds = \sqrt{[-\sin(t)]^2 + [\cos(t)]^2}\, dt = 1\, dt$$

$$\int_{\pi/2}^{3\pi/2} (2+\cos(t))(2+\sin(t)) \cdot 1 \, dt$$

$$= \int_{\pi/2}^{3\pi/2} \left[4 + 2\cos(t) + 2\sin(t) + \cos(t)\sin(t)\right] dt$$

$$= \left[4t + 2\sin(t) - 2\cos(t) + \tfrac{1}{2}\sin^2(t)\right]\Big|_{t=\pi/2}^{3\pi/2} = 4\pi - 2$$

Other Line Integrals $\quad C$ given by parametrization $(x(t), y(t))$ $a \le t \le b$

Compute $\displaystyle\int_C f(x,y)\,dx$, $\displaystyle\int_C g(x,y)\,dy$ Line integrals over C wrt dx, dy

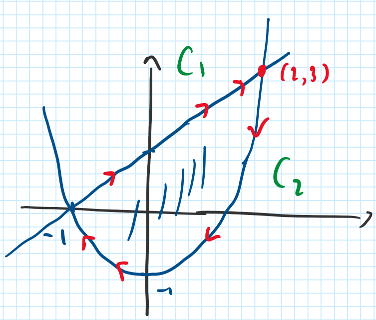
Often written as $\displaystyle\int_C f(x,y)\,dx + g(x,y)\,dy$

Note that $dx = \frac{dx}{dt}dt$, $dy = \frac{dy}{dt}dt$, so

$$\boxed{\begin{aligned} \int_C f(x,y)\,dx &= \int_a^b f(x(t),y(t))\,x'(t)\,dt \\ \int_C f(x,y)\,dy &= \int_a^b f(x(t),y(t))\,y'(t)\,dt \end{aligned}}$$

Ex. $\displaystyle\int_C y^2\,dx + 2xy\,dy$ $\quad C$ clockwise around boundary of region contained by

$$\begin{cases} y = x+1 \\ y = x^2-1 \end{cases}$$



$x^2 - 1 = y = x+1 \rightsquigarrow x^2 - 1 = x+1$

or $x^2 - x - 2 = 0$

$(x+1)(x-2) = 0$

$\boxed{x = -1, 2}$

$$\int y^2\,dx + 2xy\,dy = \int y^2\,dx + 2xy\,dy + \int_C y^2\,dx + 2xy\,dy$$

$$\int_C y^2\,dx + 2xy\,dy = \int_{C_1} y^2\,dx + 2xy\,dy + \int_{C_2} y^2\,dx + 2xy\,dy$$

C_1 $\langle -1,0\rangle \rightsquigarrow \langle 2,3\rangle$ line segment, so

$$(x(t),y(t)) = (1-t)(-1,0) + t(2,3), \quad 0\le t\le 1$$

$$= \langle t-1+2t,\ 3t\rangle = \langle 3t-1,\ 3t\rangle$$

$dx = 3\,dt$
$dy = 3\,dt$

$$\int_0^1 (3t)^2\cdot 3\,dt + 2(3t-1)\cdot 3t\cdot 3\,dt$$

$$= \int_0^1 \left[27t^2 + 6(3t-1)3t\right] dt = \int_0^1 81t^2 - 18t\,dt = 27t^3 - 9t^2\Big|_0^1$$

$$= 18$$

C_2, $x = -t$ Need to go right to left instead of left to right

$y = x^2-1 = (-t)^2 - 1 = t^2-1$

$-2\le t\le 1$ x starts at 2 goes to -1

$dx = -dt$, $dy = 2t\,dt$

$$\int_{C_2} y^2\,dx + 2xy\,dy = \int_{-2}^{1} (t^2-1)^2\cdot -1\,dt + \int_{-2}^{1} 2(-t)(t^2-1)\cdot 2t\,dt$$

$$= \int_{-2}^{1}\left[-t^4+2t^2-1-4t^4+4t^2\right] dt =$$

$$= \int_{-2}^{1} -5t^4+6t^2-1\,dt = -t^5+2t^3-t\Big|_{-2}^{1}$$

$$= 0-(32-16+2)$$

$$= -18$$

So $\int_C y^2\,dx + 2xy\,dy = \int_{C_1} y^2\,dx + 2xy\,dy + \int_{C_2} y^2\,dx + 2xy\,dy = 18-18=0!$

C

In 3D if we have a curve C parametrized by $(x(t), y(t), z(t))$, $a \le t \le b$

we have $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$

$\rho(x,y,z)$ a "density" function then

"mass" along curve is $\int_C \rho(x,y,z)\,ds = \int_a^b \rho(x,y,z) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$

Ex. C is a cylindrical spiral $\begin{cases} x(t) = \cos(t) \\ y(t) = \sin(t) \\ z(t) = t \end{cases}$ $0 \le t \le 5\pi$

$\rho(x,y,z) = y^2$ Compute $\int_C \rho\, ds$

$$\frac{dx}{dt} = -\sin(t), \quad \frac{dy}{dt} = \cos(t), \quad \frac{dz}{dt} = 1$$

$$ds = \sqrt{[-\sin(t)]^2 + [\cos(t)]^2 + 1^2} = \sqrt{2}$$

$$dx = \frac{dx}{dt}\,dt = -\sin(t)\,dt$$

$$dy = \frac{dy}{dt}\,dt = \cos(t)\,dt \qquad \rho(x,y,z) = y^2 = \sin^2(t)$$

$$dz = \frac{dz}{dt}\,dt = 1\,dt$$

$$\int_C \rho(x,y,z)\,ds = \int_0^{5\pi} \sin(t)^2 \cdot \sqrt{2}\,dt = \sqrt{2}\int_0^{5\pi} \frac{1-\cos(2t)}{2}\,dt$$

$$= \sqrt{2}\left(\frac{1}{2}t - \frac{\sin(2t)}{4}\right)\Bigg|_0^{5\pi}$$

$$\boxed{= \frac{5\sqrt{2}}{2}\pi}$$