Math 261, Lecture 30, 11/2/18

- Exam 2, Avg 67%

Today, §16.2 (finish), Next: §16.3

Recap. C a curve parametrized by $\vec{r}(t) = \langle x(t), y(t) \rangle$, $a \le t \le b$

diff of arc length $\quad ds = |\vec{r}\,'(t)|\, dt = \sqrt{(x')^2 + (y')^2}\, dt$

$* \quad |\vec{r}\,'(t)| = 1$, $a \le t \le b$ unit speed parametrization $ds = dt$

$z = f(x,y) \qquad \int_C f(x,y)\, ds = \int_a^b f(x(t), y(t))\, |\vec{r}\,'(t)|\, dt$ ↖ does not depend on parametrization

$$\int_C f(x,y)\, dx = \int_a^b f(x(t), y(t))\, x'(t)\, dt$$

$$\int_C f(x,y)\, dy = \int_a^b f(x(t), y(t))\, y'(t)\, dt$$

do depend on parametrization

$\int_C y^2\, dx + 2xy\, dy$ clockwise around region bounded by $\begin{cases} y = x+1 \\ y = x^2 - 1 \end{cases}$

$\parallel$

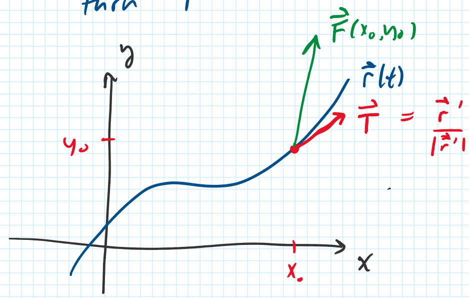
? 0!

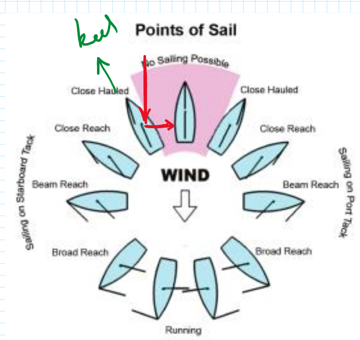
§16.2 Line Integrals of Vector Fields

$F(x,y)$ vector field

Force

particle $\vec{r}(t)$ on a path thru $\vec{F}$





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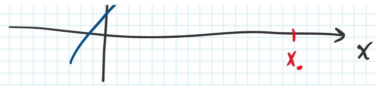
Work F is performing at a point along path



Work F is performing at a point along path

$$W = \vec{F} \cdot \vec{T}$$

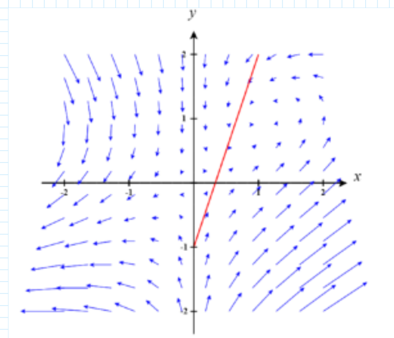
Total work performed by $\vec{F}$ along $\vec{r}(t)$

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_a^b \vec{F}(x(t), y(t)) \cdot \frac{\vec{r}'(t)}{\cancel{|\vec{r}'(t)|}} \cdot \cancel{|\vec{r}'(t)|}\, dt$$

$$\boxed{= \int_a^b \vec{F}(x(t), y(t)) \cdot \vec{r}'(t)\, dt}$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt = \int_C F \cdot d\vec{r}$$

* depends on direction of travel on curve



Ex. $\vec{F}(x,y) = \langle x - xy, x - y \rangle$

path is line from $(0,-1)$ to $(1,2)$

Compute $\int_C \vec{F} \cdot d\vec{r}$

$\vec{F}(x,y) = \langle x - xy, x - y \rangle$

$\vec{r}(t) = \langle t, 3t-1 \rangle$

$$\vec{r}(t) = (1-t)\langle 0,-1 \rangle + t\langle 1,2 \rangle, \quad 0 \le t \le 1$$

$$= \langle t, 3t-1 \rangle$$

$$\vec{r}'(t) = \langle 1, 3 \rangle$$

$$\vec{F}(x(t), y(t)) = \Big\langle t - t(3t-1),\ t - (3t-1) \Big\rangle$$

$$= \langle -3t^2 + 2t,\ -2t + 1 \rangle$$

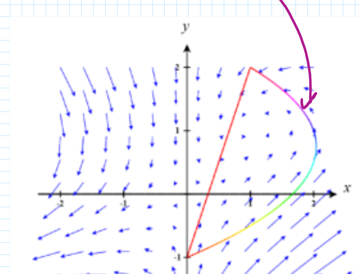
$$\vec{F} \cdot \vec{r}' = \langle -3t^2 + 2t, -2t+1 \rangle \cdot \langle 1, 3 \rangle$$

$$= -3t^2 + 2t - 6t + 3 = -3t^2 - 4t + 3$$

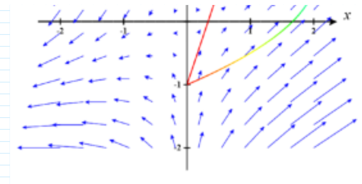
$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 -3t^2 - 4t + 3\, dt$$

$\vec{r}(t) = \langle (5+3t-2t^2)/3, t \rangle$

$-1 \le t \le 2$



Try with $\vec{r}(t) = \langle (5+3t-2t^2)/3, t \rangle \quad -1 \le t \le 2$

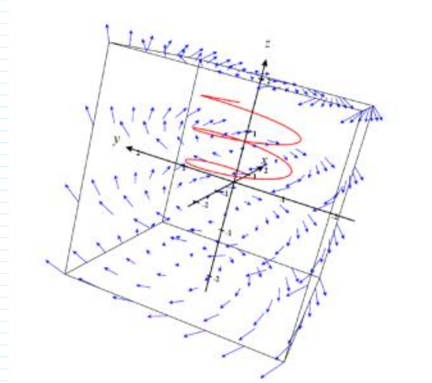


Try with $\vec{r}(t) = \langle (5+3t-2t^2)/3,\ t \rangle,\ -1 \le t \le 2$

Ex. $\vec{F}(x,y,z) = z\,i - x\,j + y\,k$

$\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle \quad 0 \le t \le 5\pi$

$$\int_C \vec{F} \cdot d\vec{r}$$



$\vec{r}\,'(t) = \langle -\sin(t), \cos(t), 1 \rangle$

$\vec{F}(\vec{r}(t)) = \langle t, -\cos(t), \sin(t) \rangle$

$\vec{F}(x,y,z) = \langle z, -x, y \rangle$

$\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = \langle t, -\cos(t), \sin(t) \rangle \cdot \langle -\sin(t), \cos(t), 1 \rangle$

$= -t\sin t - \cos^2(t) + \sin(t)$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{5\pi} -t\sin(t) \underbrace{-\cos^2(t)}_{\frac{1+\cos(2t)}{2}} + \sin(t)\, dt$$

$$= -\sin t + t\cos t - \frac{1}{2}t + \frac{\sin(2t)}{4} - \cos(t) \Big|_0^{5\pi} = -5\pi - 1 - \frac{5\pi}{2} + 2$$

$$= 1 - \frac{15\pi}{2}$$

* Work depends on end pts not choice of path when $\vec{F} = \vec{\nabla} f$ ($\vec{F}$ is conservative)

Ex. $\vec{F}(x,y,z) = yz\,i + xz\,j + xy\,k$ $\quad (\vec{F} = \vec{\nabla}(xyz))$

$\vec{r}(t) = \langle t, t^2, t^3 \rangle \quad 0 \le t \le 1$

$\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle$

$\vec{F}(\vec{r}(t)) = \langle t^2 \cdot t^3, t \cdot t^3, t \cdot t^2 \rangle = \langle t^5, t^4, t^3 \rangle$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 t^5 + 2t^5 + 3t^5 \quad dt$$

$$= \int_0^1 6t^5\, dt = t^6 \Big|_0^1 = 1$$

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What about $\vec{r}(t) = \langle t, t, \sqrt{1-t^2} - 1 + 2t \rangle$, $0 \le t \le 1$?

$$\vec{r}\,'(t) = \left\langle 1, 1, \frac{-t}{\sqrt{1-t^2}} + 2 \right\rangle$$

$$\vec{F}(\vec{r}(t)) = \left\langle t\sqrt{1-t^2} - t + 2t^2,\ t\sqrt{1-t^2} - t + 2t^2,\ t^2 \right\rangle$$

$$\vec{F}\cdot(\vec{r}(t))\cdot\vec{r}\,'(t) = 2t\sqrt{1-t^2} - 2t + 4t^2 - \frac{t^3}{\sqrt{1-t^2}} + 2t^2$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 2t\sqrt{1-t^2} + \frac{t^3}{\sqrt{1-t^2}} - 2t + 6t^2\,dt$$

$$= -\frac{2}{3}(1-t^2)^{3/2} - t^2 + 2t^3\Big|_0^1 - \int_0^1 \frac{t^3}{\sqrt{1-t^2}}\,dt$$

$u = 1-t^2$

$du = -2t\,dt$

$$\| \quad -\frac{1}{2}\int_1^0 \frac{1-u}{\sqrt{u}}$$

$$= \frac{1}{2}\left[2\sqrt{u} - \frac{2}{3}u^{3/2}\right]_0^1 = \boxed{-\frac{2}{3}}$$

$$= \frac{2}{3} - 1 + 2 - \frac{2}{3} = 1!$$

Hooray for conservative fields!!