Math 261, Lecture 31, 11/5/18

- Final Exam, Wed. 12/12, 8AM - 10AM
- Exam 2 - Avg 67%

  Cutoffs
  $A \geq 90$
  $B \geq 75$
  $C \geq 60$
  $D \geq 40$

Today: §16.3, Next: §16.4

Recap. $\vec{F}(x,y,z)$ a vector field
$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ $a \leq t \leq b$ a path C

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \underbrace{\vec{F}(\vec{r}(t))}_{\text{force acting at time } t} \cdot \vec{r}\,'(t)\, dt$$

work done by Force in direction particle is traveling

"dot product"

$$= \int_a^b \vec{F}(x(t), y(t), z(t)) \cdot \vec{r}\,'(t)\, dt$$

$\int_C \vec{F} \cdot d\vec{r}$ = Total (net) Work performed by field as particle travels path C.

$\vec{F} = \vec{\nabla} f$ then $\int_C \vec{F} \cdot d\vec{r}$ did not seem to depend on C itself, only the endpoints!

$\vec{F}$ conservative

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§ 16.3 Fundamental Theorem of Line Integrals

Try to compute $\int_C \vec{\nabla} f \cdot d\vec{r}$ along some path C.

Analogy with $\int_a^b f'(t)\,dt = f(b) - f(a)$ FTC.

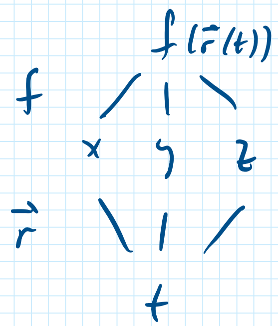
C parametrized by $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, $a \le t \le b$

$w = f(x,y,z)$, $\vec{\nabla} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$

Let's look at the integrand $\vec{\nabla} f(\vec{r}(t)) \cdot \vec{r}\,'(t)$

$$= \left\langle \frac{\partial f}{\partial x}(\vec{r}(t)), \frac{\partial f}{\partial y}(\vec{r}(t)), \frac{\partial f}{\partial z}(\vec{r}(t)) \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$$

$$= \left[\frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt} \right]$$



$$= \frac{d}{dt} f(\vec{r}(t))$$

So we conclude that $\int_C \vec{\nabla} f \cdot d\vec{r} = \int_a^b \frac{d}{dt} f(\vec{r}(t))\,dt$

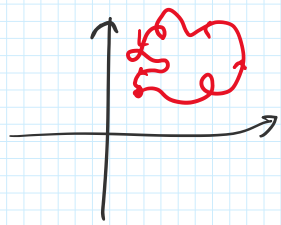
$$\underset{FTC}{=} \quad f(\vec{r}(b)) - f(\vec{r}(a))$$

Fundamental Theorem of Line Integrals

$\vec{F}$ is conservative <u>exactly</u> when

$\int_C \vec{F} \cdot d\vec{r}$ does not depend on the path chosen between any two points.

<u>closed path</u> $\vec{r}(t)$ has same start and end points
$\vec{r}(a) = \vec{r}(b)$, $a \le t \le b$



$\vec{F}$ conservative if
$\int_C \vec{F} \cdot d\vec{r} = 0$ for any closed path.

When is a vector field conservative?

$$\vec{F}(x,y) = P(x,y)\, i + Q(x,y)\, j$$

$\vec{F} = \vec{\nabla} f \qquad P = f_x$ and $Q = f_y$

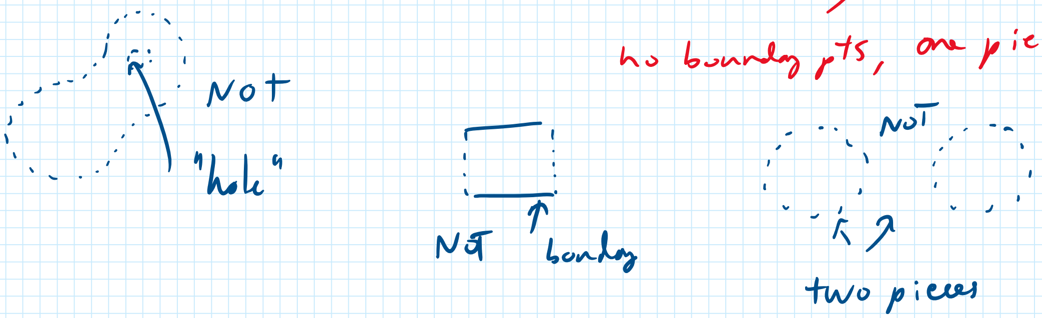
$$P_y = f_{yx} = f_{xy} = Q_x$$

$$\text{so } P_y = Q_x$$

Fact. If $P_y = Q_x$ on an <u>open, simply connected region</u>
then $\vec{F}$ is conservative

no holes

no boundary pts, one piece, no holes



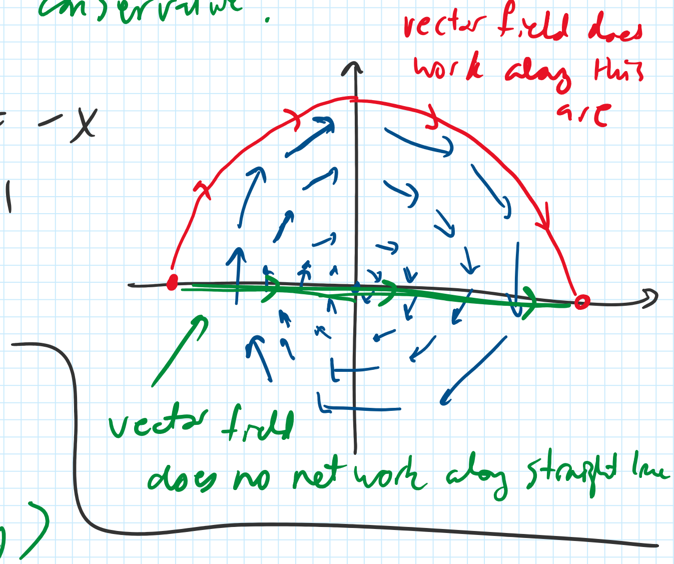
* Favorite open simply connected regions are 2D plane or 3D space.

Ex. If $\vec{F}(x,y) = y\,i - x\,j$ conservative?

$P(x,y) = y$, $Q(x,y) = -x$

$P_y = 1 \neq Q_x = -1$

so not conservative!



Ex. $\vec{F}(x,y) = \langle \overset{P}{3x^2 - 3y^2}, \overset{Q}{-6xy} \rangle$

Compute $\int_C \vec{F} \cdot d\vec{r}$ along $\vec{r}(t) = \langle e^{-t}\cos(t), e^{t}\sin(2t) \rangle$

$0 \le t \le \pi$

Is $\vec{F}$ conservative?

$P_y = -6y \overset{?}{=} Q_x = -6y$

YES!

Integrate along line from $(1,0)$ to $(-e^{-\pi}, 0)$

or, more simply, find f and evaluate at endpoints.

Let's find f. $\vec{F} = (f_x, f_y)$

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$$\begin{cases} f_x = 3x^2 - 3y^2 \\ f_y = -6xy \end{cases}$$

AD of $f_x \rightsquigarrow \overbrace{x^3 - 3xy^2 + g(y)}^{\text{guess for } f}$

$\swarrow \frac{\partial}{\partial y}$ checking my guess by diff'n y.

$$f_y = -6xy = -6xy + g'(y)$$

$g'(y) = 0$ or g is a constant, say 0

so $f = x^3 - 3xy^2$ Thus $\int_C \vec{F} \cdot d\vec{r} = f(-e^{\pi}, 0) - f(1,0)$

$$= \boxed{-e^{3\pi} - 1}$$

What about $\vec{F}(x,y,z)$?

$$\vec{F}(x,y,z) = P(x,y,z)\, i + Q(x,y,z)\, j + R(x,y,z)\, k$$

$\vec{F}$ is conservative exactly when

(P_x)

(Q_y) — (R_z)

$$\begin{cases} Q_x = P_y \\ P_z = R_x \\ Q_z = R_y \end{cases}$$

Open, simply connected region

Ex. Check that $\vec{F}(x,y,z) = (yze^{xy} + yz + y^2)\, i$
$+ (xze^{xy} + xz + 2xy)\, j$
$+ (e^{xy} + xy)\, k$

is conservative

AND find f so that $\vec{F} = \vec{\nabla} f$.

P, R $\qquad P_z = ye^{xy} + y$ ✓

$\qquad\qquad R_x = ye^{xy} + y$

AD for P in x $\rightsquigarrow$ $[ze^{xy} + xyz + xy^2] + g(y,z)$

$\swarrow \frac{\partial}{\partial y}$

$$xze^{xy} + xz + 2xy \overset{?}{=} xze^{xy} + xz + 2xy + g_y(y,z)$$

so $g_y(y,z) = 0$, or $g(y,z) = h(z)$

$\swarrow \frac{\partial}{\partial z}$

$$e^{xy} + xy \overset{?}{=} e^{xy} + xy + h'(z)$$

so $h'(z) = 0$ or h is constant.

Thus $\vec{F} = \vec{\nabla} f$ for $f(x,y,z) = ze^{xy} + xyz + xy^2 + C$

for any constant C.