Math 261, Lecture 32, 11/7/18

- Office Hours Canceled Tonight. Make up F: 4:30-5:30

Today: §16.4, Next: §16.5

Recap: Fundamental Theorem of Line Integrals

$$\int_C \vec{\nabla} f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

$\vec{r}(t)$, $a \le t \le b$ parametrizes C.

$$\vec{F}(x,y) = P(x,y)\,i + Q(x,y)\,j$$

$\vec{F} = \vec{\nabla} f$ exactly when $P_y = Q_x$

on an open, simply connected region. *

Examples. 2D plane, $x^2+y^2<1$, triangle w/o boundary, ...

§16.4 Green's Theorem

$\vec{F} = \langle P, Q \rangle$, $\vec{r}(t) = \langle x(t), y(t) \rangle$ $a \le t \le b$

parametrizes C

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \langle P, Q \rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle dt$$

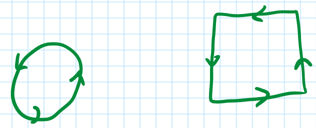
$$= \int_a^b P \frac{dx}{\cancel{dt}} \cancel{dt} + \int_a^b Q \frac{dy}{\cancel{dt}} \cancel{dt}$$

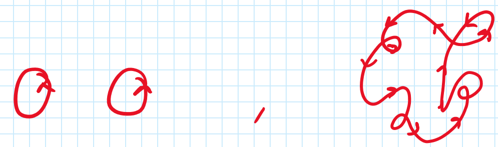
$$= \int_C P\,dx + Q\,dy$$

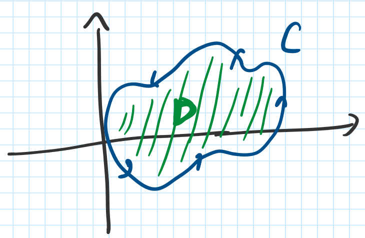
start pt = end pt

start pt = end pt

C is a simple, closed curve — one piece, "no loops"

Ex. 

Non Ex. 



C is simple, closed exactly when D is open, simply connected.

Counterclockwise — positive orientation

Clockwise — negative orientation

Green's Theorem

C simple closed curve, $\vec{F} = \langle P, Q \rangle$ vector field

D is the interior of C

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy = \iint_D Q_x - P_y \, dA$$

$$\| $$

$$\oint_C P\,dx + Q\,dy$$

$\vec{r}(t)$ goes around C once on $a \le t \le b$

Ex. $\vec{F} = \langle x - xy, x - y \rangle$, circle of radius 3 centered at origin.

$\vec{r}(t) = \langle 3\cos t, 3\sin t\rangle \quad 0 \le t \le 2\pi$ goes around circle once

$x = 3\cos t \quad dx = -3\sin t\, dt$

$y = 3\sin t \quad dy = 3\cos t\, dt$

$$\oint_C P\,dx + Q\,dy = \int_0^{2\pi} \underbrace{(3\cos t - 9\cos t\sin t)}_{P(x(t),y(t))}\underbrace{(-3\sin t)\,dt}_{dx}$$

$$+ \int_0^{2\pi} \underbrace{3(\sin t - \cos t)}_{Q(x(t),y(t))}\underbrace{3\cos t\,dt}_{dy}$$

$$\xrightarrow{\text{Green's Thm}} \iint_D Q_x - P_y\,dA$$

$$Q_x = \frac{\partial}{\partial x}(x-y) = 1$$

$$P_y = \frac{\partial}{\partial y}(x - xy) = -x$$

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_D (1+x)\,dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{3} (1 + r\cos\theta)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{9}{2} + 9\cos\theta\,d\theta = 9\pi$$

Ex. Find area inside ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$

Try polar coords ...

$$\frac{x^2}{4} + \frac{y^2}{4} - \frac{5y^2}{36}$$

$$\frac{r^2}{4} - \frac{5}{36}r^2\sin^2\theta = 1 \quad \text{or} \quad r = \frac{2}{\sqrt{1 - \frac{5}{9}\sin^2\theta}}$$

$$\text{Area} = \int_0^{2\pi}\int_0^{2/\sqrt{1-\frac{5}{9}\sin^2\theta}} r\,dr\,d\theta = \int_0^{2\pi} \frac{1}{1 - \frac{5}{9}\sin^2\theta}\,d\theta$$

NOT POSSIBLE! "elliptic integral 1st kind."

Want $\iint_D 1\,dA \qquad 1 = Q_x - P_y$?

Try: $P = -\frac{1}{2}y, \quad Q = \frac{1}{2}x$

Other possibilities: $P = -y \quad Q = 0$

$P = 0, \quad Q = x$

Want $\iint_D 1\, dA$

Try: $P = -\frac{1}{2}y$, $Q = \frac{1}{2}x$

Other possibilities: $P=-y$, $Q=0$
$P=0$, $Q=x$
$P=y$; $Q=2x$

$$\oint_C -\frac{1}{2}y\,dx + \frac{1}{2}x\,dy$$

$\vec{r}(t) = \langle 2\cos t, 3\sin t\rangle$, $0 \le t \le 2\pi$

$dx = -2\sin t\, dt$
$dy = 3\cos t\, dt$

$$\int_0^{2\pi} 3\sin^2 t\, dt + \int_0^{2\pi} 3\cos^2 t\, dt = 6\pi$$

Easy!

Ex. $\vec{F}(x,y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$

$\qquad\qquad\quad\ \ \shortparallel \qquad\qquad \shortparallel$
$\qquad\qquad\quad\ \ P \qquad\qquad Q$

$$P_y = \frac{y^2-x^2}{(x^2+y^2)^2}, \quad Q_x = \frac{y^2-x^2}{(x^2+y^2)^2}$$

But NOT conservative!

Not defined at $(0,0)$, so there is a hole.

Let's compute $\oint_C F \cdot d\vec{r} = \oint_C P\,dx + Q\,dy$

on unit circle centered at origin.

$x(t) = \cos(t)$ $\quad dx = -\sin(t)\,dt$, $0 \le t \le 2\pi$
$y(t) = \sin(t)$ $\quad dy = \cos(t)\,dt$

$$\oint P\,dx + Q\,dy = \int_0^{2\pi} \underbrace{\frac{-\sin(t)}{\cos^2(t)+\sin^2(t)}}_{P(x(t),y(t))} \cdot \underbrace{-\sin(t)\,dt}_{dx}$$

$$\underbrace{}_{P(x(t),\,y(t))} \qquad \underbrace{}_{dx}$$

$$+ \int_0^{2\pi} \underbrace{\frac{\cos(t)}{\cos^2(t)+\sin^2(t)}}_{Q(x(t),\,y(t))} \cdot \underbrace{\cos(t)\,dt}_{dy}$$

$$= \int_0^{2\pi} \sin^2(t) + \cos^2(t)\,dt = \int_0^{2\pi} 1\,dt = 2\pi!$$

This shows why $\vec{F}$ is not conservative, since it does work on a closed path.