Math 261, Lecture 33, 11/9/18

- Make up Office Hours today 4:30 - 5:30

- Office Hours Schedule

  M 3:30 - 4:30

  W 5:30 - 6:30 MATH 744

  F 11:30 - 12:30

Today §16.5, Next §16.6 (begin)

Recap: Green's Theorem

$$\vec{F}(x,y) = P(x,y)\,i + Q(x,y)\,j = \langle P, Q\rangle$$

$\oint_C \vec{F}\cdot d\vec{r}$ integral around simple, closed curve <u>once</u>, <u>counterclockwise</u>

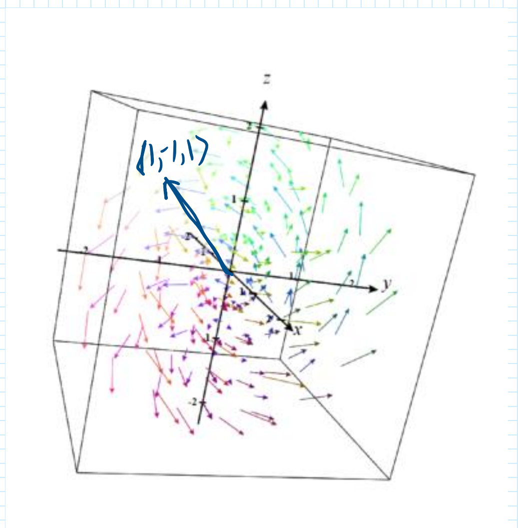
$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C P\,dx + Q\,dy \underset{\text{Green's Thm}}{=} \iint_D Q_x - P_y \, dA$$

D is region contained by C.

§16.5 Curl and Divergence

Goal - Measure how much a vector field fails to be conservative.

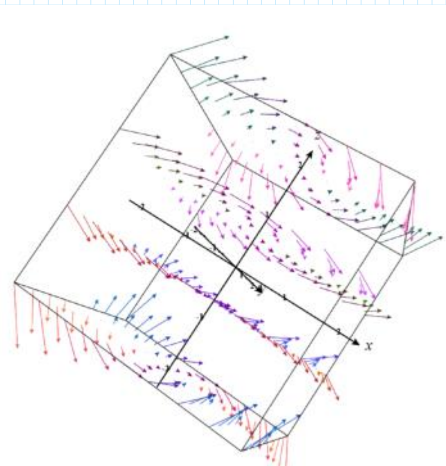
* Rotation is the thing that stops v.f. from being conservative.



In 2D $\vec{F}(x,y) = \langle P, Q\rangle$

$Q_x - P_y$ (If $= 0$, then conservative)

$\vec{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$ "del" differential operator

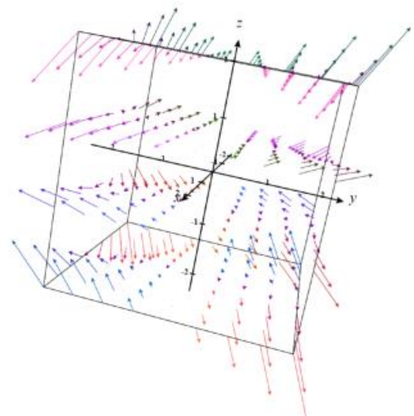


$\vec{\nabla} f$ del applied to $f(x,y,z)$, then I have gradient.

$\vec{F} = \langle P, Q, R \rangle$ $P(x,y,z)$, $Q(x,y,z)$, $R(x,y,z)$

3D vector field

$$\operatorname{curl} \vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$



$$= (R_y - Q_z)\, i + (P_z - R_x)\, j + (Q_x - P_y)\, k$$

$\vec{F}$ is conservative exactly when each of those terms is zero (if $\vec{F}$ is defined on whole 3D plane)

In other words conservative exactly when $\operatorname{curl} \vec{F} = 0$

<u>divergence</u> $\operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \langle P, Q, R \rangle$

$$= P_x + Q_y + R_z$$

outputs numbers, not vectors.

$\vec{v} \cdot (\vec{v} \times \vec{w}) = 0$

$$\operatorname{div}(\operatorname{curl} \vec{F}) = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0$$

Ex. $\vec{F} = -z\, i + x\, j + y\, k$, compute $\operatorname{curl} \vec{F}$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z & x & y \end{vmatrix} = \left(\overset{1}{\frac{\partial}{\partial y} y} - \overset{0}{\frac{\partial}{\partial z} x}\right) i$$

$$- \left(\overset{0}{\frac{\partial}{\partial x} y} - \overset{1}{\frac{\partial}{\partial z} z}\right) j$$

$$+ \left(\overset{1}{\frac{\partial}{\partial x} x} - \overset{0}{\frac{\partial}{\partial y} z}\right) k$$

$$+\left(\frac{\partial}{\partial x}\overset{1}{x}-\frac{\partial}{\partial y}\overset{0}{z}\right)k$$

$$= i - j + k = \langle 1,-1,1\rangle$$

curl $\vec{F}$ points perpendicularly to direction of rotation, using Right Hand Rule.

div curl $\vec{F} = 0$ since curl $\vec{F}$ is constant.

Ex. $\vec{F}(x,y,z) = y^2\, i + xy\, j + xyz\, k$

Compute div $\vec{F}$, curl $\vec{F}$, div curl $\vec{F}$

$$\text{div } \vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}P + \frac{\partial}{\partial y}Q + \frac{\partial}{\partial z}R$$

$$= \frac{\partial}{\partial x}y^2 + \frac{\partial}{\partial y}xy + \frac{\partial}{\partial z}xyz$$

$$= 0 + x + xy = x + xy$$

$$\text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & xy & xyz \end{vmatrix} = \left(\frac{\partial}{\partial y}xyz - \frac{\partial}{\partial z}xy\right)i$$

$$-\left(\frac{\partial}{\partial x}xyz - \frac{\partial}{\partial z}y^2\right)j$$

$$+\left(\frac{\partial}{\partial x}xy - \frac{\partial}{\partial y}y^2\right)k$$

$$= xz\, i - yz\, j - y\, k$$

$$\text{div}\cdot\text{curl } \vec{F} = \frac{\partial}{\partial x}xz + \frac{\partial}{\partial y}(-yz) + \frac{\partial}{\partial z}(-y)$$

$$= z - z + 0 = 0 \,!$$

Applications to 2D and Green's Thm.

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$$\vec{F}(x,y,z) = P(x,y)\, i + Q(x,y)\, j + 0\, k$$

$$\text{curl}\, \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = (Q_x - P_y)\, k$$

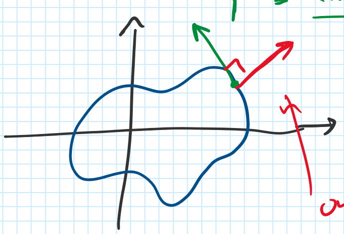
$$0 - \frac{\partial}{\partial z} Q = 0 - 0 = 0$$

Green's Thm $\displaystyle \oint_C \vec{F} \cdot d\vec{r} = \iint_D \text{curl}\, \vec{F} \cdot k \, dA$

$$\|$$

$$\oint_C \vec{F} \cdot \vec{T}\, ds$$

$$\vec{T} = \frac{\langle x'(t), y'(t) \rangle}{|\vec{r}\,'(t)|} \qquad \vec{r}(t) = \langle x(t), y(t) \rangle$$



(unit)
outward normal vector $\displaystyle \vec{n} = \frac{\langle y'(t), -x'(t) \rangle}{|\vec{r}\,'(t)|}$

Green's Thm, II

$$\oint_C \vec{F} \cdot \vec{n}\, ds = \oint_C \langle P, Q \rangle \langle dy, -dx \rangle$$

$$= \oint_C P\, dy - Q\, dx = \iint_D P_x + Q_y \, dA$$

$$\iint_D \text{div}\, \vec{F}\, dA$$

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial}{\partial x} P + \frac{\partial}{\partial y} Q$$