Math 261, Lecture 35, 11/14/18

NO CLASS Monday.

Today: §16.6 (finish), Next: §16.7 (begin)

Recap. S a surface parametrized by

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle \quad (u,v) \text{ belong to a region } D$$

Standard, Cylindrical, and Spherical Coords give us ways of parametrizing surfaces like cylinders, cones, paraboloids, spheres, planes, etc...

§16.6, Tangent Planes and Surface Area for Parametric Surfaces

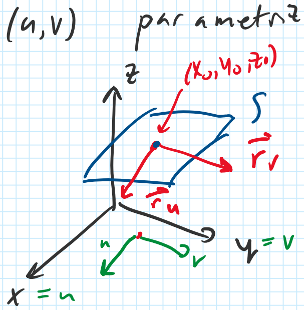
$$\vec{r}(t) = \langle x(t), y(t), z(t)\rangle \quad a \le t \le b$$

curve C

tangent vector (x_0, y_0, z_0) on C $\quad \vec{r}\,'(t_0) = \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\rangle$

$$\vec{r}(t_0) = \langle x_0, y_0, z_0\rangle$$

$\vec{r}(u,v)$ parametrizes surface S



$$\vec{r}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$$

$$\vec{r}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

$$\vec{n} = \vec{r}_u \times \vec{r}_v \quad \text{normal} \neq \vec{0}$$

"smooth"

tangent plane $\vec{n} \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$

Ex. $z \begin{cases} x = u^2 + v^2 \\ y = u^2 - v^2 \end{cases}$ $\quad (5, 3, -1)$ is on S

Find eqn for tangent plane

Ex. $\vec{r}(u,v) = \begin{cases} x = u^2 + v^2 \\ y = u^2 - v^2 \\ z = -v \end{cases}$

$(5,3,-1)$ is on S

Find eqn for tangent plane at $(5,3,-1)$

$$\vec{r}_u = \langle 2u, 2u, 0 \rangle$$
$$\vec{r}_v = \langle 2v, -2v, -1 \rangle$$

$$\vec{n} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 2u & 2u & 0 \\ 2v & -2v & -1 \end{vmatrix} = -2u\, i - (-2u)\, j - 8uv\, k$$
$$= \langle -2u, 2u, -8uv \rangle$$

Find normal $\vec{n}$ at $(5,3,-1)$

$\vec{n} = \vec{n}(u,v)$, so need to find u,v so that $\vec{r}(u,v) = \langle 5,3,-1 \rangle$

$$z = -1 = -v \rightsquigarrow v = 1$$

$v = 1$ $\qquad x = 5 = u^2 + v^2$

$u^2 = 4 \qquad u = \pm 2$

$y = 3 = u^2 - v^2$

either of $(2,1)$ or $(-2,1)$ works.

$$\vec{n}(2,1) = \langle -4, 4, -16 \rangle$$

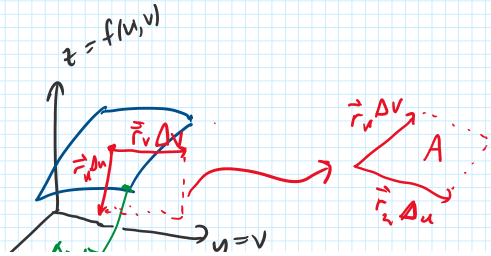
tangent plane is $\langle -4, 4, -16 \rangle \cdot \langle x-5, y-3, z+1 \rangle = 0$

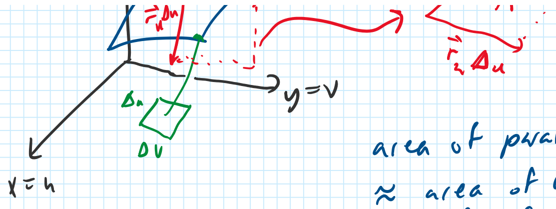
pt. $(5,3,-1)$ $\qquad \vec{r}_u(2,1) = \langle 4,4,0 \rangle$

$\vec{r}_v(2,1) = \langle 2,-2,-1 \rangle$

$$\begin{cases} x = 5 + 4u + 2v \\ y = 3 + 4u - 2v \\ z = -1 - v \end{cases}$$

Surface Area





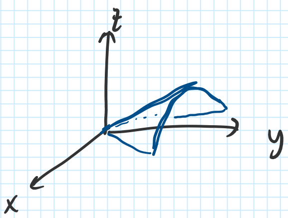
area of parallelogram
≈ area of a small piece of surface

$$A = |\vec{r}_u \Delta u \times \vec{r}_v \Delta v|$$
$$= |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v$$

$$\text{Area}(S) = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA$$

S is parametrized by (u,v) in region D.

Ex. S $\quad \vec{r}(u,v) = \begin{cases} x = 2u\cos(v) \\ y = 3u \\ z = 2u\sin(v) \end{cases} \qquad 0 \le u \le 3, \quad 0 \le v \le \pi$



half of a cone above xy plane

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 2\cos v & 3 & 2\sin v \\ -2u\sin v & 0 & 2u\cos v \end{vmatrix} = 6u\cos v \; i - \overbrace{(4u\cos^2 v + 4u\sin^2 v)}^{4u} \; j + 6u\sin v \; k$$

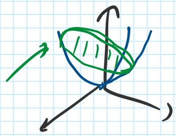
$$|\vec{r}_u \times \vec{r}_v| = \sqrt{(6u\cos v)^2 + (-4u)^2 + (6u\sin v)^2}$$
$$= \sqrt{36u^2 + 16u^2} = \sqrt{52u^2}$$
$$= \sqrt{52}\, u$$

$$\text{Area}(S) = \int_0^{\pi}\int_0^{3} \sqrt{52}\, u \; du\, dv$$
$$= \sqrt{52}\,\pi \int_0^3 u\, du = \frac{9\sqrt{52}}{2}\pi$$

Ex. Surface area of the plane $2x + 2y + z = 7$

Ex. Surface area of the plane $2x+2y+z=7$ inside of paraboloid $z=x^2+y^2$

Area.



$$2x+2y+(x^2+y^2)=7$$

$$x^2+2x+1+y^2+2y+1=7+1+1$$

$$(x+1)^2+(y+1)^2=9$$

Intersection of plane and paraboloid, which is boundary of region D.

$$\vec{r}(u,v)=\begin{cases} x=u \\ y=v \\ z=7-2u-2v \end{cases}, \quad (u+1)^2+(v+1)^2\le 9$$

$$\vec{r}_u\times\vec{r}_v=\begin{vmatrix} i & j & k \\ 1 & 0 & -2 \\ 0 & 1 & -2 \end{vmatrix}=2i-(-2)j+1k=\langle 2,2,1\rangle$$

$$|\vec{r}_u\times\vec{r}_v|=\sqrt{2^2+2^2+1^2}=3$$

$$\text{Area}(S)=\iint_D 3\,dA$$

$u+1=r\cos\theta$
$v+1=r\sin\theta$
$0\le\theta\le 2\pi$
$0\le r\le 3$

$$=\int_{\theta=0}^{2\pi}\int_{r=0}^{3} 3r\,dr\,d\theta$$

$$=\boxed{9\pi}$$