Math 261, Lecture 36, 11/16/18

- No Class Monday 11/19 and Tuesday 11/20

Today: §16.7 (begin), Next: §16.7 (finish)

Recap. S a surface parametrized by

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$$

(u,v) in a region D.

$$\text{Area}(S) = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA$$

* If $x = u$
$y = v$
$z = f(u,v)$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{1 + \left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2}$$

Point of Parametric Coords: find a nice way to describe surface *before* setting up integral rather than working in standard coords and performing a change of variables later.

§16.7 Surface Integrals of Functions

$$C \quad \vec{r}(t) = \langle x(t), y(t), z(t) \rangle \qquad a \leq t \leq b$$

$f(x,y,z)$

Line integral $$\int_C f \, ds = \int_a^b f(\vec{r}(t)) \, |\vec{r}\,'(t)| \, dt$$

$$ds = |\vec{r}\,'(t)| \, dt$$

Surface Integral.

$S \quad \vec{r}(u,v), \qquad (u,v)$ in region D

$f(x,y,z)$

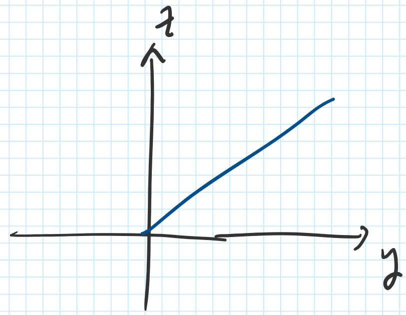
$$* \quad \boxed{\iint_S f\, dS = \iint_D f(\vec{r}(u,v))\, |\vec{r}_u \times \vec{r}_v|\, dA}$$

$$dS = |\vec{r}_u \times \vec{r}_v|\, dA$$

$*$ Does not depend on parametrization $\vec{r}(u,v)$

Ex. $\begin{cases} x = 2u\cos v \\ y = 3u \\ z = 2u\sin v \end{cases}$ $\quad 0 \le u \le 3$, $\quad 0 \le v \le \pi$

half Cone about y-axis above xy-plane



$$\iint_S x^2\, dS$$

$0 = x = 2u\cos(v)$

$\cos(v) = 0$

$v = \pi/2$

$y = 3u$

$z = 2u$

$z = \frac{2}{3}y$

$\vec{r}_u = \langle 2\cos(v), 3, 2\sin(v) \rangle$

$\vec{r}_v = \langle -2u\sin(v), 0, 2u\cos(v) \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 6u\cos v, -4u, 6u\sin v \rangle$

$|\vec{r}_u \times \vec{r}_v| = \sqrt{52}\, u$

$$\iint_S x^2\, dS = \int_{v=0}^{\pi}\int_{u=0}^{3} \underbrace{(2u\cos v)^2}_{= x^2}\ \underbrace{\sqrt{52}\, u}_{= |\vec{r}_u \times \vec{r}_v|}\ \underbrace{du\, dv}_{= dA}$$

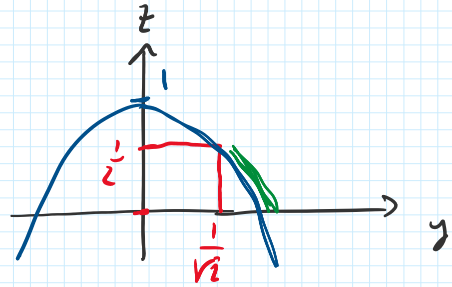
$$= 4\sqrt{2}\int_0^{\pi}\int_0^{3} u^3(\cos v)^2\,du\,dv$$

$$= 4\sqrt{2}\int_0^{\pi}\int_0^{3} u^3\left(\frac{1+\cos(2v)}{2}\right)du\,dv$$

$$= 4\sqrt{2}\cdot\frac{81}{4}\cdot\frac{1}{2}\int_0^{\pi} 1+\cos(2v)\,dv = 81\frac{\sqrt{2}}{2}\left[v+\frac{\sin(2v)}{2}\right]\Bigg|_0^{\pi} = 81\frac{\sqrt{2}}{2}\pi$$

Ex. S is a paraboloid $z = 1-x^2-y^2$, $0 \le z \le \frac{1}{2}$

$$\iint_S x^2+y^2\,dS$$



$$\vec{r}(u,v) = \begin{cases} x = u\cos(v) \\ y = u\sin(v) \\ z = 1-u^2 \end{cases}$$

$0 \le v \le 2\pi$

$\frac{1}{\sqrt{2}} \le u \le 1$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ \cos(v) & \sin(v) & -2u \\ -u\sin(v) & u\cos(v) & 0 \end{vmatrix}$$

$$= \left\langle 2u^2\cos(v), -2u^2\sin(v), \underbrace{u\cos^2 v + u\sin^2 v}_{u}\right\rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{\underbrace{4u^4\cos^2 v + 4u^4\sin^2 v}_{=4u^4} + u^2}$$

$$= \sqrt{4u^4+u^2} = u\sqrt{4u^2+1}$$

$$x^2+y^2 = (u\cos(v))^2 + (u\sin(v))^2 = u^2$$

$$\iint_S x^2+y^2\, dS = \int_{v=0}^{2\pi}\int_{u=\frac{1}{\sqrt{2}}}^{1} u^2 \cdot u\sqrt{4u^2+1}\; du\,dv$$

$$w = 4u^2+1$$
$$dw = 8u\; du$$
$$u^2 = \frac{w-1}{4}$$

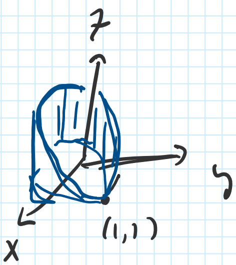
$$= \int_{v=0}^{2\pi}\int_{w=3}^{5} \frac{w-1}{4}\cdot\sqrt{w}\cdot\frac{1}{8}\, dw\, dv$$

Using standard coords we can parametrize this surface as:

$$\begin{cases} x = u \\ y = v \\ z = 1-u^2-v^2 \end{cases} \qquad \left(\frac{1}{\sqrt{2}}\right)^2 \leq u^2+v^2 \leq 1$$

$$\vec{r}_u \times \vec{r}_v = \langle 2u, 2v, 1\rangle$$

Ex. $\iint_S z\, dS$ S is on the cylinder $x^2+y^2=2$ between $z=0$ and $z = 2-x-y$



$$\vec{r}(u,v) = \begin{cases} x = \sqrt{2}\cos(v) \\ y = \sqrt{2}\sin(v) \\ z = u \end{cases}$$

$0 \le v \le 2\pi$

$0 \le u \le 2 - \sqrt{2}\cos v - \sqrt{2}\sin v$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ -\sqrt{2}\sin v & \sqrt{2}\cos v & 0 \end{vmatrix} = \langle -\sqrt{2}\cos v, \sqrt{2}\sin v, 0 \rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{(-\sqrt{2}\cos v)^2 + (\sqrt{2}\sin v)^2 + 0^2}$$

$$= \sqrt{2+2+0} = 2$$

$$\iint_S z \, dS = \int_{v=0}^{2\pi} \int_{u=0}^{2-\sqrt{2}\cos v - \sqrt{2}\sin v} u \cdot 2 \, du \, dv$$

$$= \int_0^{2\pi} \left(2 - \sqrt{2}\cos v - \sqrt{2}\sin v\right)^2 dv$$

$$= \int_0^{2\pi} \Big(4 + 2\cos^2 v + 2\sin^2 v - 4\sqrt{2}\cos v - 4\sqrt{2}\sin v + \\ + 4\cos v \sin v \Big) dv$$

$$= \int_0^{2\pi} 8 - 4\sqrt{2}\cos v - 4\sqrt{2}\sin v + 4\cos v \sin v \, dv$$

$$= \Big[8v - 4\sqrt{2}\sin v + 4\sqrt{2}\cos v + 2\sin^2 v \Big]^{2\pi}$$

$$= \Big[8v - 4\sqrt{2}\sin v + 4\sqrt{2}\cos v + 2\sin^2 v \Big]_0$$

$$= 8\pi$$