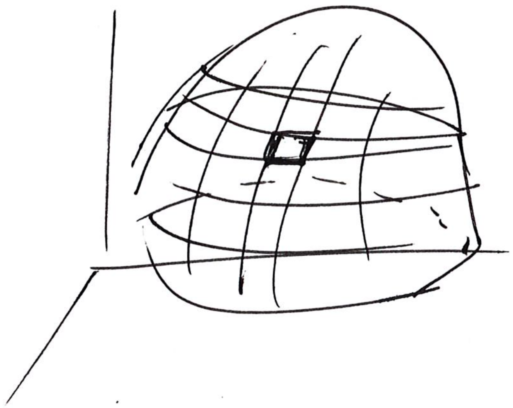
Section 16.7 (Cont.) - Flux Integrals.

Review - Surface Integrals



parametrize S by

$r(u,v)$

$= \left(x(u,v),\ y(u,v),\ z(u,z) \right)$

for (u,v) in D.

$$\iint_S f(x,y,z)\, dS$$

$$= \iint_D f(r(u,v))\, |\vec{r}_u \times \vec{r}_v| \underbrace{dA}_{du\,dv}$$

where $\vec{r}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$

$\vec{r}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$

Flux Integrals

vector field $\vec{F}$ in $\mathbb{R}^3$

compute flux of $\vec{F}$ through surface S.

Simple example

$\vec{F}$ = rate at which rain is falling

$\vec{F} = \langle 0, 0, -10 \rangle$

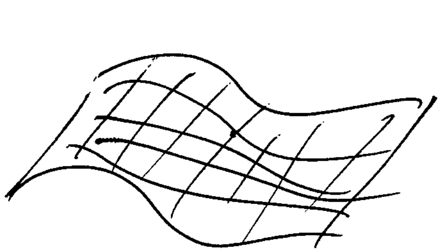
$S = [0,1] \times [0,1]$ in x-y plane

rainfall hits S at rate $\left(\begin{smallmatrix}\text{rainfall}\\ \text{rate}\end{smallmatrix}\right) \times (\text{Area})$

harder F is more complicated (wind)

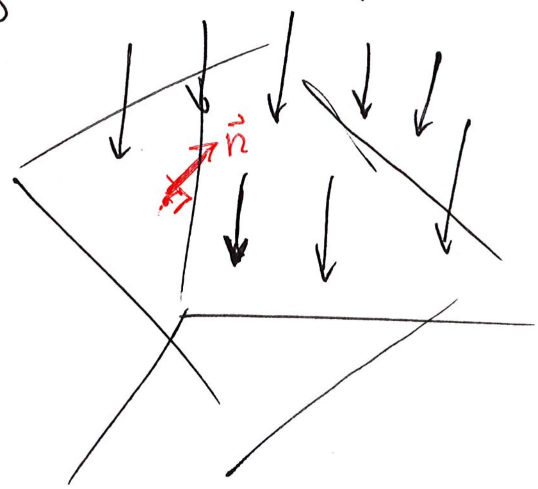
S is some curved surface.

idea



compute flux through tiny "rectangles"

To compute flux of a constant vector field ~~#~~ $\vec{F}$ through a tilted plane.



$\vec{n}$ be unit normal for plane.

$\vec{F} \cdot \vec{n}$

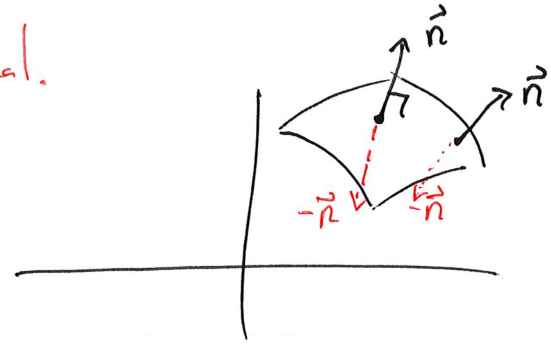
<u>flux</u> $(\vec{F} \cdot \vec{n})(\text{Area})$

If S is a surface parameterized by $\vec{r}(u,v)$ the unit normal at point $\vec{r}(\bar{u},v)$ is given by

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

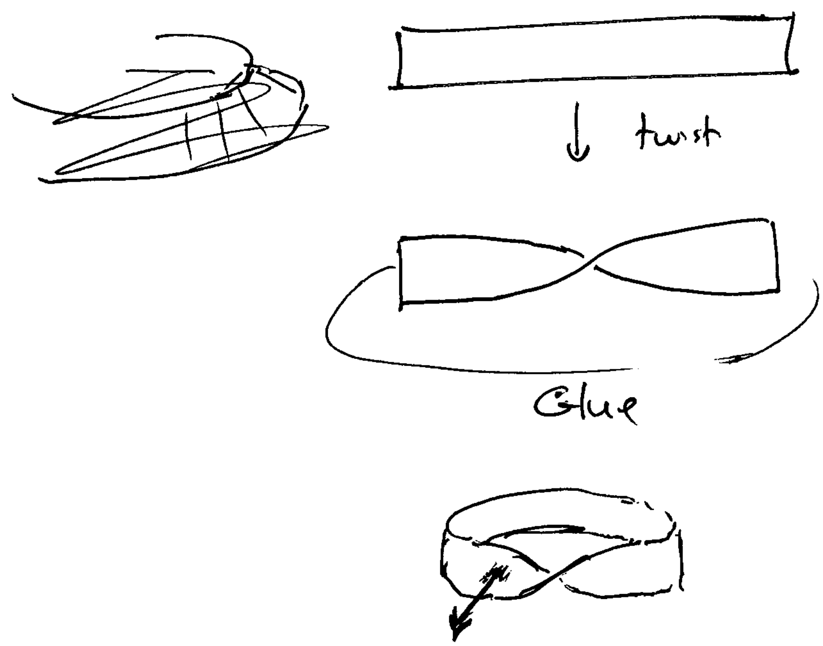
<u>note</u> $$-\vec{n} = \frac{-(\vec{r}_u \times \vec{r}_v)}{|\vec{r}_u \times \vec{r}_v|} = \frac{\vec{r}_v \times \vec{r}_u}{|\vec{r}_v \times \vec{r}_u|}$$

is also a unit normal.

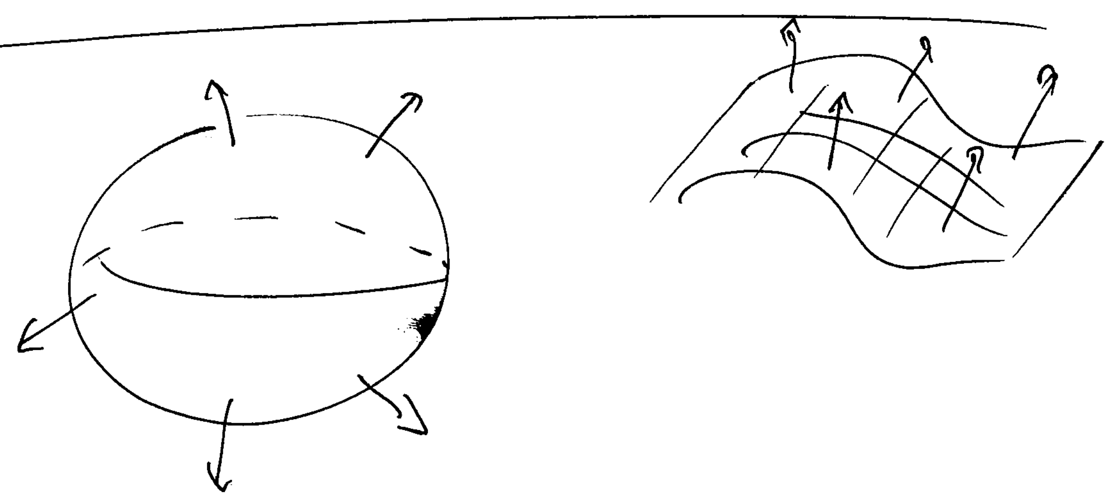


Def: A ~~surf~~ parametrized surface is orientable if we can define unit normal at each point in a continuous way.

Counter-example (non-orientable surface) Möbius Strip.



orientable



Flux of $\vec{F} = \vec{F}(x, y, z)$

through orientable surface S.
parameterized by $\vec{r}(u, v)$.

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S (\vec{F} \cdot \vec{n})\, dS$$

$$= \iint_D \vec{F}(\vec{r}(u,v)) \cdot \vec{n}(\vec{r}(u,v)) \cdot |\vec{r}_u \times \vec{r}_v|\, dA$$

$$= \iint_D \vec{F}(\vec{r}(u,v)) \cdot \frac{\vec{r}_u \times \vec{r}_v}{\cancel{|\vec{r}_u \times \vec{r}_v|}} \cancel{|\vec{r}_u \times \vec{r}_v|}\, dA$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, dA$$

$$\iint_S \vec{F} \cdot d\vec{S}$$

$$= \iint_D \vec{F}(\vec{r}(u,v)) \cdot \overset{(\vec{r}_u \times \vec{r}_v)}{\cancel{\vec{n}(\vec{r}(u,v))}} \cdot dA$$

$$= \int_0^{\pi}\int_0^{3} \langle 2u\sin(v),\ 2u\cos(v),\ 3u \rangle \cdot \langle 6u\cos(v), -4u, 6u\sin(v) \rangle \, du\, dv$$

$$= \int_0^{\pi}\int_0^{3} \left(12u^2 \sin(v)\cos(v) - 8u^2\cos(v) + 18u^2 \sin(v)\right) du \, dv$$

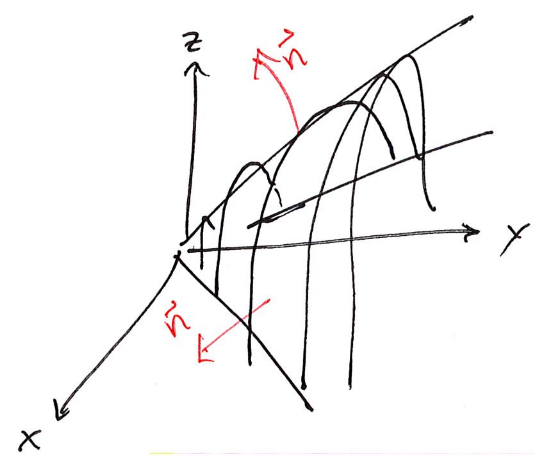
= Some number.

Example $\vec{F}(x,y,z) = z\vec{i} + x\vec{j} + y\vec{k}$

$= \langle z, x, y \rangle$

Parameterized Surface.

$\vec{r}(u,v) = \langle 2u\cos(v), 3u, 2u\sin(v) \rangle$ $\quad 0 \le u \le 3$, $0 \le v \le \pi$



half-cone above x-y plane

need an orientation "outward" pointing normal

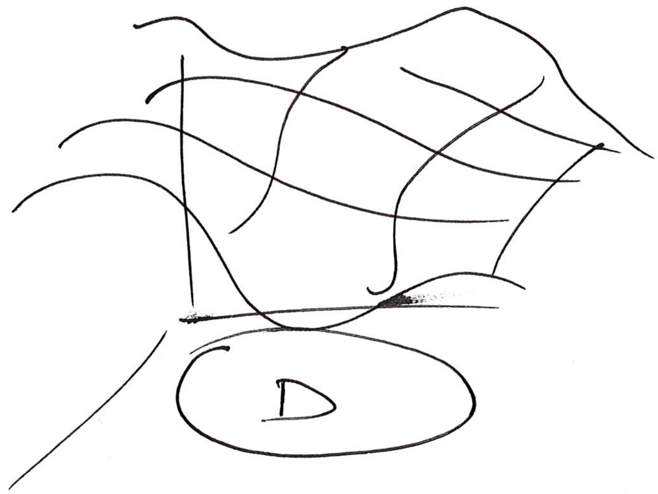
$\vec{r}_u = \langle 2\cos(v), 3, 2\sin(v) \rangle$

$\vec{r}_v = \langle -2u\sin(v), 0, 2u\cos(v) \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 6u\cos(v), \underline{-4u}, 6u\sin(v) \rangle$

this is the "outward" normal.

__Special Case__ Surface is portion of graph $z = g(x,y)$



$\cancel{\vec{r}(u,v) = \langle u, v, g(u,v)\rangle}$ $\quad \vec{r}(u,v) = \langle u, v, g(u,v)\rangle$

__upward normal__ $\quad \vec{n} = \left\langle -\frac{\partial g}{\partial u}, \frac{\partial g}{\partial v}, 1 \right\rangle$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot \left\langle -\frac{\partial g}{\partial u}, \frac{\partial g}{\partial v}, 1 \right\rangle dA$$

$\vec{F} = \langle P, Q, R \rangle$

$$= \iint_D \left(P\left(-\frac{\partial g}{\partial u}\right) + Q\frac{\partial g}{\partial v} + R \right) du\, dv$$