Math 261, Lecture 37, 11/26/18

Today §16.7 (finish), Next: §16.8 (all)

Recap: S a surface parametrized by $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$
(u,v) in a region D

$$\vec{r}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle, \quad \vec{r}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

For $f(x,y,z)$ a real valued function the <u>Surface Integral</u>

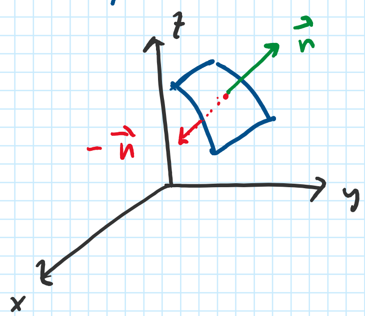
$$\iint_S f(x,y,z)\, dS = \iint_D f(\vec{r}(u,v)) \underbrace{|\vec{r}_u \times \vec{r}_v|\, dA}_{dS}$$

§16.7 (cont'd) Surface Integrals of Vector Fields

S a parametric surface given by $\vec{r}(u,v)$

the <u>unit normal</u> to S at a point is given by

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$



We have $-\vec{n} = \dfrac{\vec{r}_v \times \vec{r}_u}{\quad}$

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So we can choose $\vec{n}$ to <u>point</u> <u>outward</u> (<u>positive</u>)

$\vec{n}(u,v)$ is in same direction as $\vec{r}(u,v)$

<u>or</u> $\vec{n}$ can <u>point</u> <u>inward</u> (<u>negative</u>)

$\vec{n}(u,v)$ is in opposite direction of $\vec{r}(u,v)$

S is <u>orientable</u> if $\vec{n}$ can be made positive always

spheres, cones, cylinders, paraboloids are all orientable

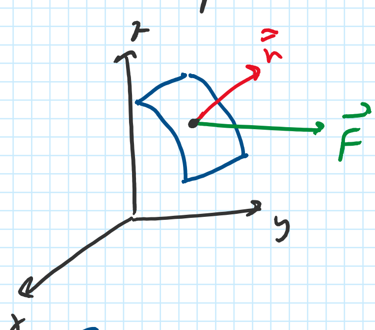
möbius band is <u>not</u> (see Stewart)

Let $\vec{F}(x,y,z)$ be a vector field

Component of $\vec{F}$ "passing thru" surface at a point

is given by component in direction of (positive)

unit normal $\vec{F}\cdot\vec{n}$



The flux is the total flow thru S

The <u>flux</u> is the total flow thru S

$$\text{Flux} = \iint_S \vec{F}\cdot\vec{n}\ dS$$

Notationally,

$$\text{Flux} = \iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\ dS = \iint_D \vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\ dA$$

We use this formula to compute flux.

Ex. $\vec{F}(x,y,z) = z\,i + x\,j + y\,k = \langle z, x, y\rangle$

$$\vec{r}(u,v) = \begin{cases} x = 2u\cos v \\ y = 3u \\ z = 2u\sin v \end{cases} \qquad \begin{matrix} 0 \le u \le 3 \\ 0 \le v \le \pi \end{matrix}$$

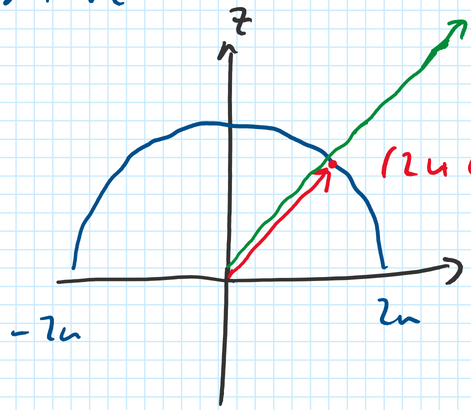
"half cone above xy plane"

Compute $\iint_S \vec{F}\cdot d\vec{S}$ for positive orientation.

$\vec{r}_u = \langle 2\cos v, 3, 2\sin v\rangle$, $\vec{r}_v = \langle -2u\sin v, 0, 2u\cos v\rangle$

$$\vec{r}_u \times \vec{r}_v = \langle 6u\cos v, -4u, 6u\sin v\rangle$$

positive? Look at slice (xz-plane)



point in same direction so positive!

$$\vec{F}(\vec{r}(u,v)) = \langle 2u\sin v, 2u\cos v, 3u \rangle$$

$$\vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) = 12u^2 \sin v \cos v - 8u^2 \cos v + 18u^2 \sin v$$

$$\iint_S \vec{F} \cdot d\vec{S} = \int_{v=0}^{\pi} \int_{u=0}^{3} 12u^2 \sin v \cos v - 8u^2 \cos v + 18u^2 \sin v \, du \, dv$$

Ex. $\vec{F}(x,y,z) = xy\, i + yz\, j + xz\, k$

S given by $\vec{r}(u,v) = \begin{cases} x = u \\ y = v \\ z = 4 - u^2 - v^2 \end{cases}$ $\quad -1 \le u \le 1$, $-1 \le v \le 1$

Compute $\iint_S \vec{F} \cdot d\vec{S}$

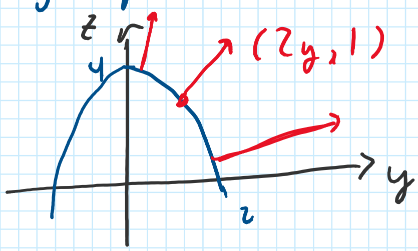
$\vec{r}_u = \langle 1, 0, -2u \rangle$

$\vec{r}_v = \langle 0, 1, -2v \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 2u, 2v, 1 \rangle$ positive?

Since S is part of a paraboloid symmetric about z-axis

look at yz-plane $(y = v)$



positive z-value so points outward, thus positive

Another way: $\langle y, z \rangle = \langle y, 4 - y^2 \rangle$ when $x = 0$

$\langle y, 4-y^2 \rangle \cdot \langle 2y, 1 \rangle = 2y^2 + 4 - y^2 = y^2 + 4 > 0$

so $\langle 2y, 1 \rangle$ points in same direction as $\langle y, z \rangle$

$$\vec{F}(\vec{r}(u,v)) = \langle uv, v(4-u^2-v^2), u(4-u^2-v^2) \rangle$$

$$= \langle uv, 4v - u^2v - v^3, 4u - u^3 - uv^2 \rangle$$

$$\vec{F} \cdot (\vec{r}_u \times \vec{r}_v) = 2u^2v + (8v^2 - 2u^2v - 2v^4) + (4u - u^3 - uv^2)$$

$$\iint_S \vec{F} \cdot d\vec{S} = \int_{-1}^{1}\int_{-1}^{1} (4u + 8v^2 - u^3 + 2u^2v - uv^2 - 2u^3v - 2v^4)\,du\,dv$$