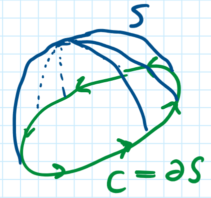
Math 261, Lecture 39, 11/30/18

Today: §11.9 (all), Next: REVIEW

Recap: Stoke's Theorem

C a simple, closed curve which is boundary a surface S

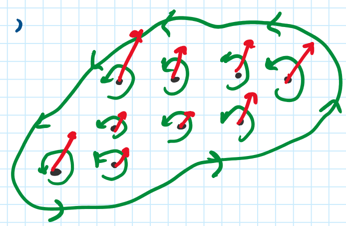
$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$$



Sometimes write C as ∂S "boundary of surface S"

C positively oriented (counterclockwise), S positively oriented (outward)

Intuition: Let C be a closed curve. $\oint_C \vec{F} \cdot d\vec{r}$ represents circulation around C, which is the "sum" of all circulations around each point.



Now curl captures the circulation around each point. That is curl is the normal to the direction of rotation of the vector field.

Note that if $\vec{F} = \langle P, Q \rangle = \langle P, Q, 0 \rangle$ and S is in the xy plane, so $\begin{cases} x = u \\ y = v \\ z = 0 \end{cases}$ (u,v) in D parametrizes S

We have $\operatorname{curl} \vec{F} = \langle 0, 0, Q_x - P_y \rangle$

$\vec{r}_u \times \vec{r}_v \, dA = \langle 0, 0, 1 \rangle \, dA$

So $\iint \operatorname{curl} \vec{F} \cdot d\vec{S} = \iint \operatorname{curl} \vec{F} \cdot \vec{r}_u \times \vec{r}_v \, dA$

S $\qquad\qquad\qquad\qquad$ S

$$= \iint_D Q_x - P_y \, dA$$

Thus Green's Thm is a special case of Stoke's Thm.

§16.9 The Divergence Theorem

E a <u>simple solid</u> region in 3D space
"One piece, no holes"

boundary of E, $\partial E = S$ an orientable surface

E is a region trapped between a combination of planes, cylinders, spheres, paraboloids, cones, etc.

Divergence Theorem (Gauss's Theorem)

S is a surface bounding a simple solid region E

$$\boxed{\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV}$$

S is positively oriented (outward normal)

Recall if $\vec{F} = (P, Q, R)$ $\operatorname{div} \vec{F} = P_x + Q_y + R_z$

Intuition: The divergence $\operatorname{div} \vec{F}$ captures the resistance to flow through a point in space. Thus the total flow (flux) through a region E, given by total flow (flux) through a region E, given by

$\iint_S \vec{F} \cdot d\vec{S}$ is the "sum" of the total

resistance to the field flow through each

point in E $\iiint_E \operatorname{div} \vec{F} \, dV$

Note that $\operatorname{div} \vec{F}$ is a number valued function, <u>NOT</u> a vector field, so there is no "dot" in the triple integral.

Ex. $\vec{F} = (x^3 - z^3)\,i + (y^3 + z^3)\,j + z^3\,k$

Compute $\iint_S \vec{F} \cdot d\vec{S}$ where S is the

hemisphere above the xy plane of radius 2

and the disk of radius 2 in the xy plane at the origin.

We have $\iint_S \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S}$

parametrizing the hemisphere S_1 and computing $\iint_{S_1} \vec{F} \cdot d\vec{S}$ is complicated

so let's simplify by computing $\iiint_E \operatorname{div} \vec{F} \, dV$ instead.

$$\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 3x^2 + 3y^2 + 3z^2$$

so $\iiint_E \operatorname{div} \vec{F} \, dV = 3 \iiint_E x^2 + y^2 + z^2 \, dV$ over the part of ball of radius 2 above xy plane.

Use spherical coordinates:

$\rho^2 = x^2+y^2+z^2$, $0 \le \rho \le 2$

$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi/2$ since E lies above xy-plane ($\phi = \pi/2$)

By Divergence Thm

$$\iint_S \vec{F}\cdot d\vec{S} = 3\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2}\int_{\rho=0}^{2} \rho^2\,\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

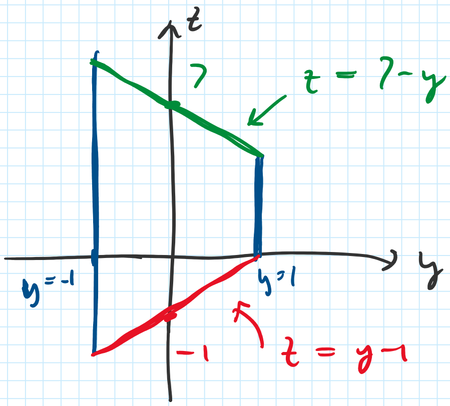
$$= 3\int_0^{2\pi}\int_0^{\pi/2}\int_0^2 \rho^4 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \frac{3}{5}\cdot 32 \cdot 2\pi \cdot \int_0^{\pi/2} \sin\phi \, d\phi$$

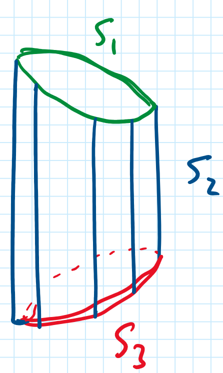
$$= \frac{192}{5}\pi$$

Ex. $\vec{F} = x^3 y \, i + x^3 z \, j + 3y^3 z \, k$

Compute $\iint_S \vec{F}\cdot d\vec{S}$ over surface S whose side is the cylinder $x^2+y^2=1$, whose top is $x+y+z=7$ and whose bottom is $x+y-z=1$



so



To compute need to split into 3 integrals for each piece of surface

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S} + \iint_{S_3} \vec{F} \cdot d\vec{S}$$

or compute $\iiint_E \operatorname{div} \vec{F} \, dV$ for region E bounded by S

Which sounds better? :)

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(x^3 y) + \frac{\partial}{\partial y}(x^3 z) + \frac{\partial}{\partial z}(3y^3 z)$$

$$= 3x^2 y + 0 + 3y^3$$

$$= 3y(x^2 + y^2)$$

Let's use cylindrical coordinates to describe E

$$r^2 = x^2 + y^2, \quad 0 \le r \le 1$$

$$0 \le \theta \le 2\pi$$

$$x + y - 1 \le z \le 7 - x - y$$

$$r\cos\theta + r\sin\theta - 1 \le z \le 7 - r\cos\theta - r\sin\theta$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV = \iiint_E 3y(x^2+y^2) \, dV$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=r\cos\theta+r\sin\theta-1}^{7-r\cos\theta-r\sin\theta} 3r\cos\theta (r^2) \cdot r \, dz \, dr \, d\theta$$

$$\theta=0 \qquad r=0 \qquad z=r\cos\theta+r\sin\theta-1$$

$$= \int_0^{2\pi}\int_0^1 3r^4\cos\theta\left(\left(7-r\cos\theta-r\sin\theta\right)-\left(r\cos\theta+r\sin\theta-1\right)\right)dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 3r^4\cos\theta\left(8-2r\cos\theta-2r\sin\theta\right)dr\,d\theta$$

$$= 24\int_0^{2\pi}\int_0^1 r^4\cos\theta\,dr\,d\theta - 6\int_0^{2\pi}\int_0^1 r^5\cos^2\theta\,dr\,d\theta - 6\int_0^{2\pi}\int_0^1 r^5\cos\theta\sin\theta\,dr\,d\theta$$

$$= 0 - \int_0^{2\pi}\cos^2\theta\,d\theta - \int_0^{2\pi}\cos\theta\sin\theta\,d\theta$$

$$= -\int_0^{2\pi}\left(\frac{1}{2}+\frac{1}{2}\cos(2\theta)\right)d\theta - \frac{1}{2}\sin^2\theta\,\Big|_0^{2\pi}$$

$$= -\pi + 0 = -\pi$$