Math 261, Lecture 6, 8/31/18

Today: §13.2 (all)

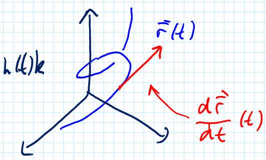
Reminder, NO CLASS Monday 9/3 (Labor Day Holiday)

§13.2 Derivatives and Integrals of Vector Functions

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$x = f(t)$
$y = g(t)$
$z = h(t)$

$\vec{r}(t) = f(t)\,i + g(t)\,j + h(t)\,k$



$\frac{d\vec{r}}{dt} = \langle f'(t), g'(t), h'(t) \rangle$

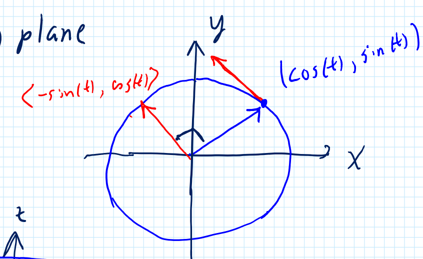
derivative = tangent direction to curve

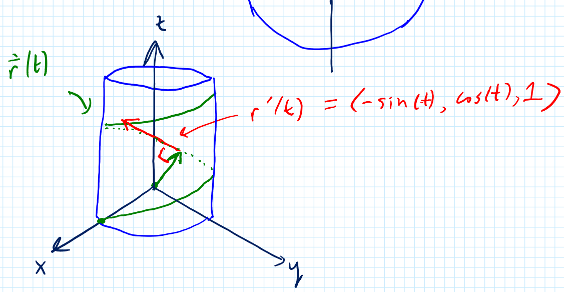
Ex. Cylindrical Spiral

$x = \cos(t)$	$x' = -\sin(t)$
$y = \sin(t)$	$y' = \cos(t)$
$z = t$	$z' = 1$

x', y': xy plane

Look at xy plane





Let's find the tangent line to $\vec{r}(t)$ at $t = \pi/3$

tangent line = pt on curve + tangent direction at that pt

$t = \pi/3$ $\quad \vec{r}(\pi/3) = (\cos(\pi/3), \sin(\pi/3), \pi/3)$
$= (1/2, \sqrt{3}/2, \pi/3)$

$\vec{r}\,'(\pi/3) = (-\sin(\pi/3), \cos(\pi/3), 1)$

tangent line is $\langle 1/2, \sqrt{3}/2, \pi/3 \rangle + t \langle -\sqrt{3}/2, 1/2, 1 \rangle$

$x = 1/2 - \sqrt{3}/2\, t$
$y = \sqrt{3}/2 + 1/2\, t$
$z = \pi/3 + t$

$\boxed{\text{Unit tangent vector at } t \text{ is } = \vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}}$

$\vec{v}$ unit vector is same direction is $\dfrac{1}{|\vec{v}|}\vec{v}$

Ex. $\vec{r}(t) = \langle 3\sin(t),\, e^t,\, 3\cos(t)\rangle$

$\vec{r}\,'(t) = \langle 3\cos(t),\, e^t,\, -3\sin(t)\rangle$

$$|\vec{r}\,'(t)| = \sqrt{(3\cos(t))^2 + (e^t)^2 + (-3\sin(t))^2}$$

$$= \sqrt{\underline{9\cos(t)^2} + e^{2t} + \underline{9\sin(t)^2}}$$

$$= \sqrt{9+e^{2t}}$$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \left\langle \frac{3\cos(t)}{\sqrt{9+e^{2t}}},\, \frac{e^t}{\sqrt{9+e^{2t}}},\, -\frac{3\sin(t)}{\sqrt{9+e^{2t}}}\right\rangle$$

$t \to \infty \quad \langle 0,\, 1,\, 0\rangle$

Rules of Differentiation

- for each component normal diff rules

$\vec{u}(t),\ \vec{v}(t)$

The dot product is a number valued function of t.

$$\frac{d}{dt}\left[\vec{u}(t)\cdot\vec{v}(t)\right] = \vec{u}\,'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}\,'(t)$$

$$\frac{d}{dt}\left[\vec{u}(t)\times\vec{v}(t)\right] = \vec{u}\,'(t)\times v(t) + \vec{u}(t)\times\vec{v}\,'(t)$$

don't switch order!

Ex. $\vec{u}(t) = \langle \sin(t), \cos(t), 1\rangle$
$\vec{v}(t) = \langle e^t, e^{2t}, e^{3t}\rangle$

$\vec{u}(t)\cdot\vec{v}(t) = e^t\sin(t) + e^{2t}\cos(t) + e^{3t}$

$$\vec{u}(t)\times\vec{v}(t) = \begin{vmatrix} i & j & k \\ \sin(t) & \cos(t) & 1 \\ e^t & e^{2t} & e^{3t}\end{vmatrix}$$

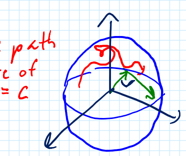
$$= \begin{vmatrix}\cos(t) & 1 \\ e^t & e^{3t}\end{vmatrix} i - \begin{vmatrix}\sin(t) & 1 \\ e^t & e^{3t}\end{vmatrix} j + \begin{vmatrix}\sin(t) & \cos(t) \\ e^t & e^{2t}\end{vmatrix} k$$

$$= \langle \cos(t)e^{3t} - e^t,\, -\sin(t)e^{3t} + e^t,\, \sin(t)e^{2t} - \cos(t)e^t\rangle$$

Usually easier to differentiate $\vec{u}, \vec{v}$ then take cross product

Ex. $|\vec{r}(t)| = c$

$\vec{r}(t)$ is a path on sphere of radius $= c$



then $\vec{r}(t) \perp \vec{r}\,'(t)$

(See Example 4 p. 858 in book)

Integration

$\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$

$\vec{R}(t) = \langle F(t), G(t), H(t)\rangle$ is an antiderivative to $\vec{r}(t)$ if

$F' = f$
$G' = g$
$H' = h$

FTC $$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt,\, \int_a^b g(t)\,dt,\, \int_a^b h(t)\,dt\right\rangle$$

$$= \vec{R}(b) - \vec{R}(a)$$

Ex. $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$, $\int_0^{\pi/2} \vec{r}(t)\, dt$

$$\vec{R}(t) = \left\langle \sin(t) + C_1, -\cos(t) + C_2, \frac{t^2}{2} + C_3 \right\rangle$$

$$= \left\langle \sin(t), -\cos(t), \frac{t^2}{2} \right\rangle + \langle C_1, C_2, C_3 \rangle$$

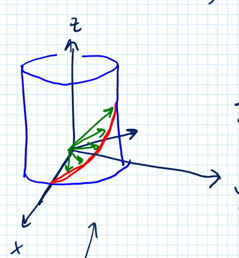
$\hookrightarrow$ AD is unique up to constant vector $\vec{C} = \langle C_1, C_2, C_3 \rangle$

$$\int_0^{\pi/2} \vec{r}(t)\, dt = \vec{R}(\pi/2) - \vec{R}(0)$$

$$= \left\langle \sin(\pi/2), -\cos(\pi/2), \frac{\left(\frac{\pi}{2}\right)^2}{2} \right\rangle - \left\langle \sin(0), -\cos(0), \frac{0^2}{2} \right\rangle$$

$$= \langle 1 - 0, 0 - (-1), \pi^2/8 - 0 \rangle$$

$$= \langle 1, 1, \pi^2/8 \rangle$$



the black arrow (integral) is the "sum" of the green arrows

$\int_0^{\pi/2} \vec{r}(t)\, dt$ = "net direction of travel (displacement) on curve from $t=0$ to $t=\pi/2$"

$$\frac{1}{b-a} \int_a^b \vec{r}(t)\, dt = \frac{\vec{R}(b) - \vec{R}(a)}{b-a}$$

Average direction of travel.